Institut Teknologi Sepuluh Nopember
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DEPARTEMEN TEKNIK FISIKA - FTIRS

OPERATOR E – EKSPEKTASI pada FUNGSI VARIABEL ACAK

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Fs. VARIABEL ACAK

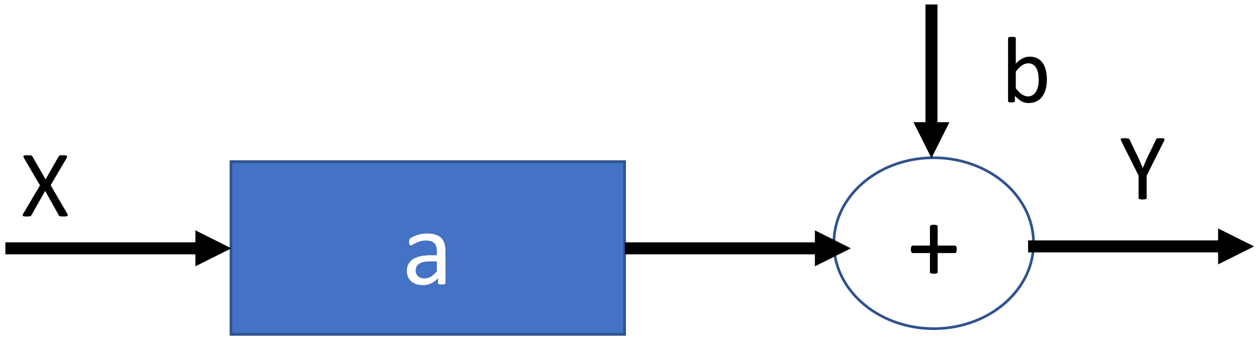
Capaian Pembelajaran:

Mampu menentukan karakteristik dari fungsi Variabel Acak

Kajian:

1. **Fungsi Variabel acak**
2. **Karakteristik Fungsi Variabel Acak**

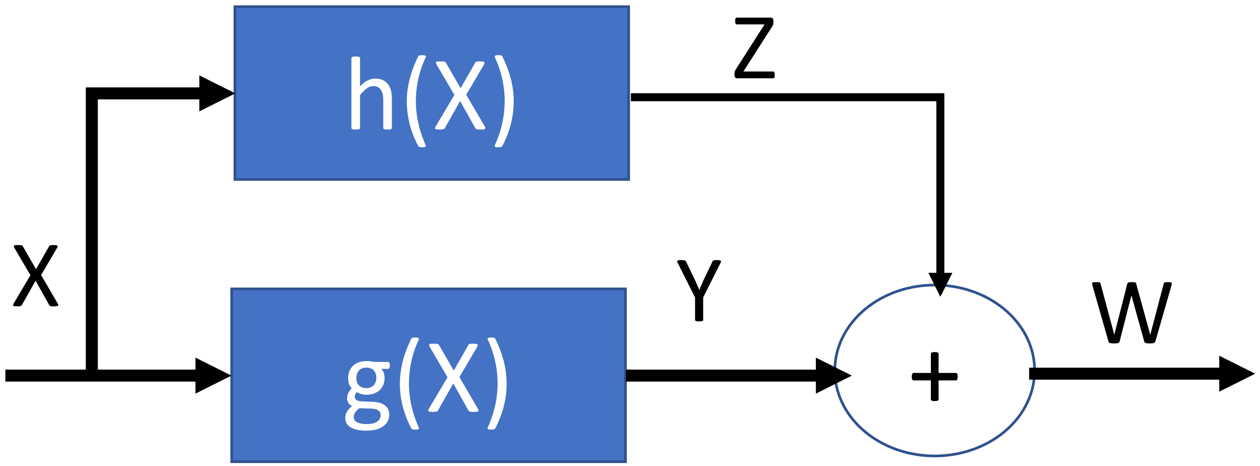
Operator E dikenakan pada aX + b



Theorem 4.5: If a and b are constants, then

$$E(aX + b) = aE(X) + b.$$

Operator E dikenakan pada jumlahan fungsi dari Variabel Acak

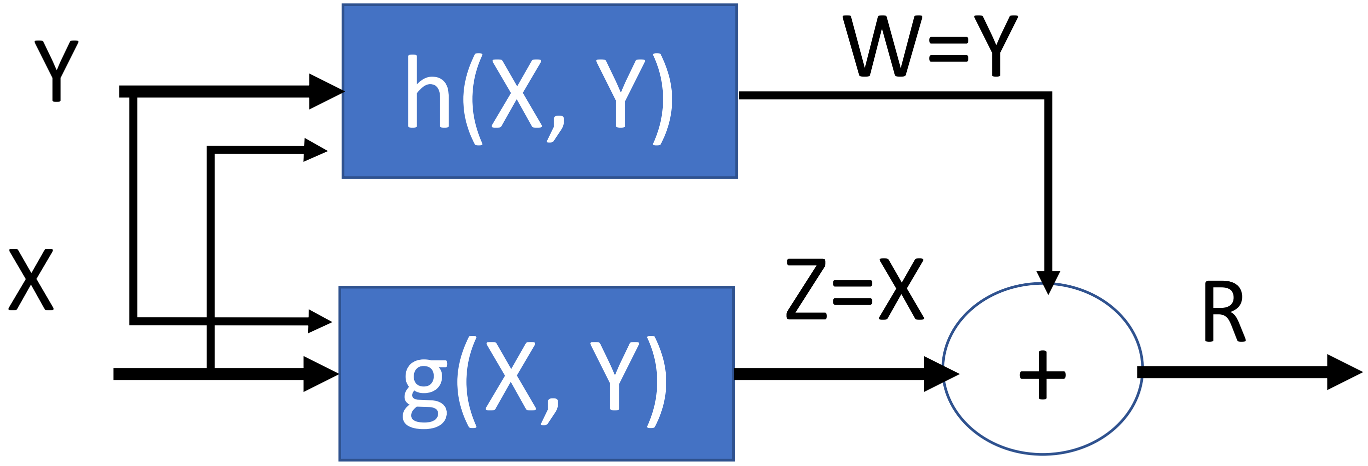


W = Y + Z

Theorem 4.6: The expected value of the sum or difference of two or more functions of a random variable X is the sum or difference of the expected values of the functions. That is,

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)].$$

Perhatikan bila g(X,Y) = X; dan h(X,Y) = Y



R = Z + W
= X + Y

Corollary 4.4: Setting $g(X,Y) = X$ and $h(X,Y) = Y$, we see that

$$E[X \pm Y] = E[X] \pm E[Y].$$

Bila X dan Y dua variabel acak independen, maka berlaku:

Theorem 4.8: Let X and Y be two independent random variables. Then

$$E(XY) = E(X)E(Y).$$

Dan besarnya kovarian

Corollary 4.5: Let X and Y be two independent random variables. Then $\sigma_{XY} = 0$.

Theorem 4.9: If X and Y are random variables with joint probability distribution $f(x, y)$ and a, b, and c are constants, then

$$\sigma^2_{aX+bY+c} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY}.$$

***Proof*:** By definition, $\sigma^2_{aX+bY+c} = E\{[(aX + bY + c) - \mu_{aX+bY+c}]^2\}$. Now

$$\mu_{aX+bY+c} = E(aX + bY + c) = aE(X) + bE(Y) + c = a\mu_X + b\mu_Y + c,$$

by using Corollary 4.4 followed by Corollary 4.2. Therefore,

$$\begin{aligned}\sigma^2_{aX+bY+c} &= E\{[a(X - \mu_X) + b(Y - \mu_Y)]^2\}\\ &= a^2E[(X - \mu_X)^2] + b^2E[(Y - \mu_Y)^2] + 2abE[(X - \mu_X)(Y - \mu_Y)]\\ &= a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY}.\end{aligned}$$

Corollary 4.6: Setting $b = 0$, we see that

$$\sigma^2_{aX+c} = a^2\sigma^2_X = a^2\sigma^2.$$

Corollary 4.7: Setting $a = 1$ and $b = 0$, we see that

$$\sigma^2_{X+c} = \sigma^2_X = \sigma^2.$$

Corollary 4.8: Setting $b = 0$ and $c = 0$, we see that

$$\sigma^2_{aX} = a^2\sigma^2_X = a^2\sigma^2.$$

Corollary 4.9: If X and Y are independent random variables, then

$$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y.$$

The result stated in Corollary 4.9 is obtained from Theorem 4.9 by invoking Corollary 4.5.

Corollary 4.10: If X and Y are independent random variables, then

$$\sigma^2_{aX-bY} = a^2\sigma^2_X + b^2\sigma^2_Y.$$

Corollary 4.10 follows when b in Corollary 4.9 is replaced by $-b$. Generalizing to a linear combination of n independent random variables, we have Corollary 4.11.

Corollary 4.11: If $X_1, X_2, \ldots, X_n$ are independent random variables, then

$$\sigma^2_{a_1X_1+a_2X_2+\cdots+a_nX_n} = a_1^2\sigma^2_{X_1} + a_2^2\sigma^2_{X_2} + \cdots + a_n^2\sigma^2_{X_n}.$$

Example 4.22: If X and Y are random variables with variances $\sigma_X^2 = 2$ and $\sigma_Y^2 = 4$ and covariance $\sigma_{XY} = -2$, find the variance of the random variable $Z = 3X - 4Y + 8$.

Solution:

$$\begin{aligned} \sigma_Z^2 = \sigma_{3X-4Y+8}^2 &= \sigma_{3X-4Y}^2 && \text{(by Corollary 4.6)} \\ &= 9\sigma_X^2 + 16\sigma_Y^2 - 24\sigma_{XY} && \text{(by Theorem 4.9)} \\ &= (9)(2) + (16)(4) - (24)(-2) = 130. \end{aligned}$$

Example 4.23: Let X and Y denote the amounts of two different types of impurities in a batch of a certain chemical product. Suppose that X and Y are independent random variables with variances $\sigma_X^2 = 2$ and $\sigma_Y^2 = 3$. Find the variance of the random variable $Z = 3X - 2Y + 5$.

Solution:

$$\begin{aligned} \sigma_Z^2 = \sigma_{3X-2Y+5}^2 &= \sigma_{3X-2Y}^2 && \text{(by Corollary 4.6)} \\ &= 9\sigma_x^2 + 4\sigma_y^2 && \text{(by Corollary 4.10)} \\ &= (9)(2) + (4)(3) = 30. \end{aligned}$$

Untuk fungsi Variabel cak yang nonlinier

$$g(x) = g(\mu_X) + \left.\frac{\partial g(x)}{\partial x}\right|_{x=\mu_X} (x - \mu_X) + \left.\frac{\partial^2 g(x)}{\partial x^2}\right|_{x=\mu_X} \frac{(x - \mu_X)^2}{2} + \cdots .$$

Approximation of $E[g(X)]$

$$E[g(X)] \approx g(\mu_X) + \left.\frac{\partial^2 g(x)}{\partial x^2}\right|_{x=\mu_X} \frac{\sigma_X^2}{2}.$$

Example 4.24: Given the random variable X with mean μ_X and variance σ_X^2, give the second-order approximation to $E(e^X)$.

***Solution*:** Since $\frac{\partial e^x}{\partial x} = e^x$ and $\frac{\partial^2 e^x}{\partial x^2} = e^x$, we obtain $E(e^X) \approx e^{\mu_X}(1 + \sigma_X^2/2)$.

Similarly, we can develop an approximation for $\mathrm{Var}[g(x)]$ by taking the variance of both sides of the first-order Taylor series expansion of $g(x)$.

Approximation of $\mathrm{Var}[g(X)]$

$$\mathrm{Var}[g(X)] \approx \left[\frac{\partial g(x)}{\partial x}\right]^2_{x=\mu_X} \sigma_X^2.$$

Latihan

4.67 If the joint density function of X and Y is given by

$$f(x,y) = \begin{cases} \frac{2}{7}(x+2y), & 0 < x < 1,\ 1 < y < 2, \\ 0, & \text{elsewhere}, \end{cases}$$

find the expected value of $g(X,Y) = \frac{X}{Y^3} + X^2Y$.

4.68 The power P in watts which is dissipated in an electric circuit with resistance R is known to be given by $P = I^2R$, where I is current in amperes and R is a constant fixed at 50 ohms. However, I is a random variable with $\mu_I = 15$ amperes and $\sigma_I^2 = 0.03$ amperes^2. Give numerical approximations to the mean and variance of the power P.

4.85 Suppose it is known that the life X of a particular compressor, in hours, has the density function

$$f(x) = \begin{cases} \frac{1}{900}e^{-x/900}, & x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the mean life of the compressor.

(b) Find $E(X^2)$.

(c) Find the variance and standard deviation of the random variable X.

4.91 A dealer's profit, in units of \$5000, on a new automobile is a random variable X having density function

$$f(x) = \begin{cases} 2(1-x), & 0 \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the variance of the dealer's profit.

(b) Demonstrate that Chebyshev's theorem holds for $k = 2$ with the density function above.

(c) What is the probability that the profit exceeds \$500?

4.98 A convenience store has two separate locations where customers can be checked out as they leave. These locations each have two cash registers and two employees who check out customers. Let X be the number of cash registers being used at a particular time for location 1 and Y the number being used at the same time for location 2. The joint probability function is given by

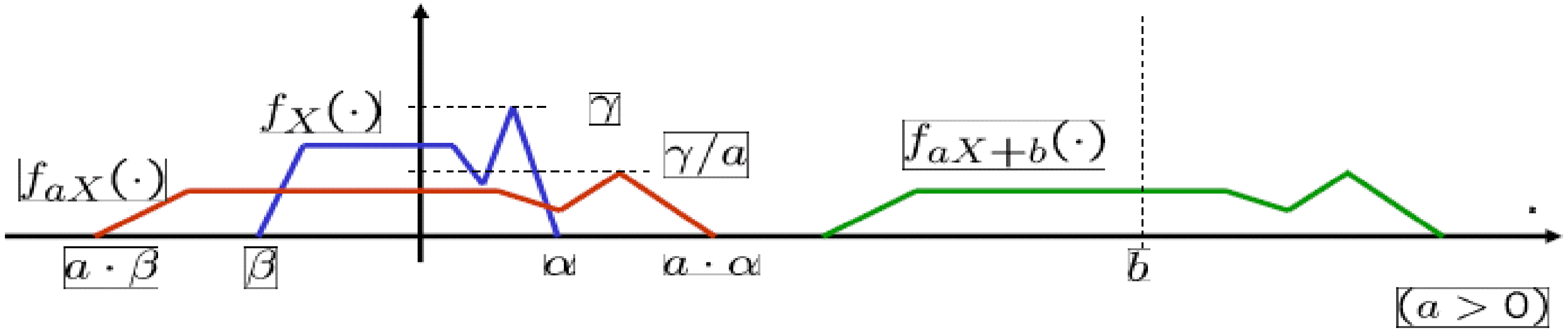
	y		
x	0	1	2
0	0.12	0.04	0.04
1	0.08	0.19	0.05
2	0.06	0.12	0.30

(a) Give the marginal density of both X and Y as well as the probability distribution of X given $Y = 2$.

(b) Give $E(X)$ and $\text{Var}(X)$.

(c) Give $E(X \mid Y = 2)$ and $\text{Var}(X \mid Y = 2)$.

The PDF of $Y = aX + b$.

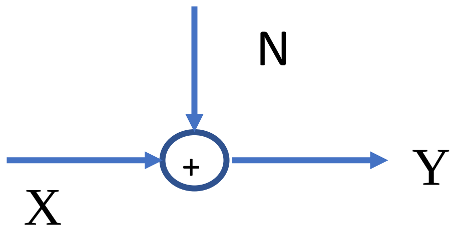


$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

- Use this to check that if X is normal, then $Y = aX + b$ is also normal.

Latihan

1. Berikut ini merupakan diagram sebuah sinyal Y hasil dari sinyal sumber X dan sinyal noise N, apabila X berdistribusi Normal, dengan rata-rata 100 dan standard dev 5, sedangkan noise berdistribusi normal dengan rata-rata 10 dan standard deviasi 2.



a. Tetukan Rata – rata dan variansi dari sinyal Y
b. Probabilitas sinyal Y bernilai kurang dari 105
c. Probabilitas sinyal Y bernilai lebih dari 110
d. Probabilitas sinyal Y kurang dari 105 pada saat N bernilai kurang dari 5

Latihan

2. Apakah pernyataan berikut Benar atau Salah:
a. Sebuah variabel acak X diskrit selalu mempunyai sifat fungsi distribusi kerapatan probabilitas nya diskrit
b. Sebuah variabel acak jamak X selalu mempunyai fungsi distribusi kerapatan probabilitas f_X yang bernilai > 0
c. Variabel acak jamak X dan Y selalu mempunyai fungsi distribusi gabungan yang nilainya selalu > 1

Distribusi Prob. Bersyarat

- Probabilitas kejadian A dengan syarat kejadian B didefinisikan:

$$P\{A|B\} = \frac{P\{A \cap B\}}{P\{B\}}$$

- CDF bersyarat:

$$F\{x|B\} = F_X(x|B\} = P\{X \leq x|B\} = \frac{P\{(X \leq x) \cap B\}}{P\{B\}}$$

- PDF bersyarat:

$$f(x|B) = \frac{dF(x|B)}{dx} = \frac{\frac{d}{dx}P\{X \leq x, B\}}{P\{B\}}$$

- Semua sifat CDF berlaku juga untuk CDF bersyarat
- Semua sifat PDF berlaku juga untuk PDF bersyarat

Terimakasih