**Commutator of \bgroup\color{black}$E$\egroupand \bgroup\color{black}$t$\egroup**

Again use the crutch of keeping a wave function on the right to avoid mistakes.

\begin{eqnarray*}[E,t]\psi(x,t)&=&\left(i\hbar{\partial\over \partial t}t-ti\hba...
...rtial\over \partial t}\right)\psi(x,t)\\
&=&i\hbar\psi(x,t)\\
\end{eqnarray*}

Removing the wave function, we have the commutator.

\begin{displaymath}\bgroup\color{black} [E,t]=i\hbar \egroup\end{displaymath}

**Commutator of \bgroup\color{black}$E$\egroupand \bgroup\color{black}$x$\egroup**

Again use the crutch of keeping a wave function on the right to avoid mistakes.

\begin{eqnarray*}[E,x]\psi(x,t)&=&\left(i\hbar{\partial\over \partial t}x-xi\hba...
...ial t}-i\hbar x{\partial\over \partial t}\right)\psi(x,t) =0 \\
\end{eqnarray*}

Since.



**Commutator of \bgroup\color{black}$L_x$\egroupand \bgroup\color{black}$L_y$\egroup**

Angular momentum is defined by

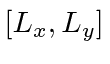
\begin{displaymath}\bgroup\color{black} \vec{L}=\vec{r}\times\vec{p} .\egroup\end{displaymath}

So the components of angular momentum are

\begin{displaymath}\bgroup\color{black} L_z=xp_y-yp_x \egroup\end{displaymath}

\begin{displaymath}\bgroup\color{black} L_x=yp_z-zp_y \egroup\end{displaymath}

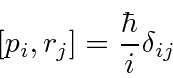
\begin{displaymath}\bgroup\color{black} L_y=zp_x-xp_z .\egroup\end{displaymath}

We wish to compute which has all the coordinates and momenta in it.

The only operators that do not commute are the coordinates and their conjugate momenta.

\begin{displaymath}\bgroup\color{black} [x,y]=0 \egroup\end{displaymath}

\begin{displaymath}\bgroup\color{black} [p_x,p_y]=0 \egroup\end{displaymath}

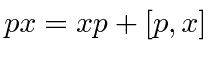


So now we just need to compute.

\begin{eqnarray*}[L_x,L_y]&=&[yp_z-zp_y,zp_x-xp_z]\\
&=&[yp_z,zp_x]-[yp_z,xp_z...
...-0-0+x[z,p_z]p_y\\
&=&{\hbar\over i}(yp_x-xp_y)=i\hbar L_z \\
\end{eqnarray*}

It is not necessary (or wise) to use the differential operators and a wave function crutch to compute commutators like this one. **Use the known basic commutators when you can.**

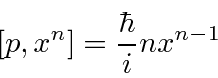
**Commutator of \bgroup\color{black}$p$\egroupand \bgroup\color{black}$x^n$\egroup**

We can use the commutator to help us. Remember that 

\begin{eqnarray*}[p,x^n]&=&px^n-x^np\\
&=&(px)x^{n-1}-x^np\\
&=&xpx^{n-1}+[p...
...p,x]x^{n-1}-x^np\\
&=&n[p,x]x^{n-1}=n{\hbar\over i}x^{n-1} \\
\end{eqnarray*}

It is usually not wise to use the differential operators and a wave function crutch to compute commutators like this one. **Use the known basic commutators when you can.**Nevertheless, we can compute it that way.

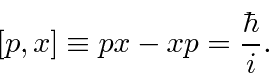
\begin{displaymath}\bgroup\color{black}[p,x^n]\psi={\hbar\over i}{\partial\over\...
...{\partial\over\partial x}\psi={\hbar\over i}nx^{n-1}\psi\egroup\end{displaymath}



It works pretty well for this particular case, but not if I have \bgroup\color{black}$p$\egroupto some power...

## Commutators

## Operators (or variables in quantum mechanics) do not necessarily commute. We can compute the [commutator](http://quantummechanics.ucsd.edu/ph130a/130_notes/node109.html#section:commutator) of two variables, for example



Later we will learn to derive the uncertainty relation for two variables from their commutator. We will also use commutators to solve several important problems.

## Commutators of \bgroup\color{black}$A$\egroup, \bgroup\color{black}$A^\dagger$\egroupand \bgroup\color{black}$H$\egroup

We will use the commutator between \bgroup\color{black}$A$\egroupand \bgroup\color{black}$A^\dagger$\egroupto solve the HO problem. The operators are defined to be

\begin{eqnarray*}
A &=&\left(\sqrt{m\omega\over2\hbar}x+i{p\over\sqrt{2m\hbar\om...
...t{m\omega\over2\hbar}x-i{p\over\sqrt{2m\hbar\omega}}\right) .\\
\end{eqnarray*}

The **commutator** is

\begin{eqnarray*}[A,A^\dagger]&=&{m\omega\over2\hbar}[x,x]+{1\over 2m\hbar\omega...
...\\
&=&{i\over 2\hbar}(-[x,p]+[p,x])={i\over \hbar}[p,x]=1 .\\
\end{eqnarray*}

Lets use this simple commutator

\begin{displaymath}\bgroup\color{black} [A,A^\dagger]=1 \egroup\end{displaymath}

to compute **commutators with the Hamiltonian**. This is easy if \bgroup\color{black}$H$\egroupis written in terms of \bgroup\color{black}$A$\egroupand \bgroup\color{black}$A^\dagger$\egroup.

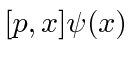
\begin{eqnarray*}[H,A]&=&\hbar\omega[A^\dagger A,A]=\hbar\omega[A^\dagger,A]A=-\...
...er]=\hbar\omega A^\dagger[A,A^\dagger]=\hbar\omega A^\dagger \\
\end{eqnarray*}

## Commutators

Operators (or variables in quantum mechanics) do not necessarily commute. We can see our first example of that now that we have a few operators. We **define the commutator** to be

|  |
| --- |
| \bgroup\color{black}$\displaystyle [p,x]\equiv px-xp$\egroup |

(using \bgroup\color{black}$p$\egroupand \bgroup\color{black}$x$\egroupas examples.)

We will now **compute the commutator** between \bgroup\color{black}$p$\egroupand \bgroup\color{black}$x$\egroup. Because \bgroup\color{black}$p$\egroupis represented by a differential operator, we must do this carefully. Lets think of the commutator as a (differential) operator too, as generally it will be. To make sure that we keep all the \bgroup\color{black}${\partial\over\partial x}$\egroupthat we need, we will compute then remove the at the end to see only the commutator.

\begin{eqnarray*}[p,x]\psi(x)&=&px\psi(x)-xp\psi(x)
={\hbar\over i}{\partial\ove...
...\partial\psi(x)\over\partial x}\right)
={\hbar\over i}\psi(x)\\
\end{eqnarray*}

So, removing the we used for computational purposes, we get the commutator.

|  |
| --- |
| \bgroup\color{black}$\displaystyle [p,x]={\hbar\over i}$\egroup |

Later we will learn to derive the uncertainty relation for two variables from their commutator. Physical variable with zero commutator have no uncertainty principle and we can know both of them at the same time.

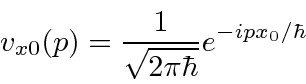
We will also use commutators to solve several important problems.

We can compute the **same commutator in momentum space**.

\begin{eqnarray*}[p,x]\phi&=&[p,i\hbar{d\over dp}]\phi=i\hbar\left(p{d\over dp}\...
...=i\hbar(-\phi)={\hbar\over i}\phi\\
{[p,x]}&=&{\hbar\over i}\\
\end{eqnarray*}  
The commutator is the same in any representation.

## Operators in Momentum Space

If we want to work in **momentum space**, we need to look at the states of definite position to find our operators. The state (in momentum space) with definite position \bgroup\color{black}$x_0$\egroupis



The operators are

|  |
| --- |
| \bgroup\color{black}$\displaystyle x^{(op)}=i\hbar{\partial\over \partial p}$\egroup |

and

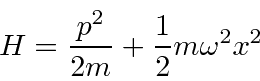
\begin{displaymath}\bgroup\color{black}p^{(op)}=p .\egroup\end{displaymath}

The \bgroup\color{black}$(op)$\egroupnotation used above is usually dropped. If we see the variable \bgroup\color{black}$p$\egroup, use of the operator is implied (except in state written in terms of \bgroup\color{black}$p$\egrouplike ).

# Harmonic Oscillator Solution using Operators

Operator methods are very useful both for solving the Harmonic Oscillator problem and for any type of computation for the HO potential. The operators we develop will also be useful in quantizing the electromagnetic field.

The Hamiltonian for the **1D Harmonic Oscillator**



looks like it could be written as the square of a operator. It can be rewritten in terms of [the operator \bgroup\color{black}$A$\egroup](http://quantummechanics.ucsd.edu/ph130a/130_notes/node168.html#section:HOoper)

|  |
| --- |
| \bgroup\color{black}$\displaystyle A\equiv \left(\sqrt{m\omega\over2\hbar}x+i{p\over\sqrt{2m\hbar\omega}}\right)$\egroup |

and its Hermitian conjugate \bgroup\color{black}$A^\dagger$\egroup.

|  |
| --- |
| \bgroup\color{black}$\displaystyle H=\hbar\omega\left(A^\dagger A+{1\over 2}\right) $\egroup |

We will use the [commutators](http://quantummechanics.ucsd.edu/ph130a/130_notes/node169.html#section:HOcom) between \bgroup\color{black}$A$\egroup, \bgroup\color{black}$A^\dagger$\egroupand \bgroup\color{black}$H$\egroupto solve the HO problem.

|  |
| --- |
| \bgroup\color{black}$\displaystyle [A,A^\dagger]=1 $\egroup |

The commutators with the Hamiltonian are easily computed.

\begin{eqnarray*}[H,A]&=&-\hbar\omega A \\
{[H,A^\dagger]}&=&\hbar\omega A^\dagger \\
\end{eqnarray*}

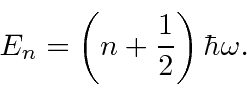
From these commutators we can show that \bgroup\color{black}$A^\dagger$\egroupis a [raising operator](http://quantummechanics.ucsd.edu/ph130a/130_notes/node170.html#section:HOraise) for Harmonic Oscillator states

|  |
| --- |
| \bgroup\color{black}$\displaystyle A^\dagger u_n=\sqrt{n+1}u_{n+1} $\egroup |

and that \bgroup\color{black}$A$\egroupis a **lowering operator**.

|  |
| --- |
| \bgroup\color{black}$\displaystyle Au_n=\sqrt{n}u_{n-1}$\egroup |

Because the lowering must stop at a ground state with positive energy, we can show that the allowed energies are



The [actual wavefunctions](http://quantummechanics.ucsd.edu/ph130a/130_notes/node173.html#section:HOwavefn) can be deduced by using the differential operators for \bgroup\color{black}$A$\egroupand \bgroup\color{black}$A^\dagger$\egroup, but often it is more useful to define the \bgroup\color{black}$n^{th}$\egroupeigenstate in terms of the ground state and raising operators.

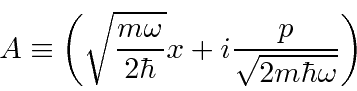
|  |
| --- |
| \bgroup\color{black}$\displaystyle u_n={1\over\sqrt{n!}}(A^\dagger)^nu_0 $\egroup |

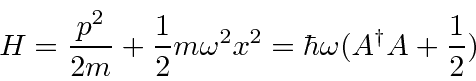
Almost **any calculation** of interest can be done without actual functions since we can express the operators for position and momentum.

\begin{eqnarray*}
x&=&\sqrt{\hbar\over 2m\omega}(A+A^\dagger) \\
p&=&-i\sqrt{m\hbar\omega\over 2}(A-A^\dagger) \\
\end{eqnarray*}

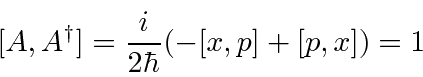
## Harmonic Oscillator Solution with Operators

We can solve the [harmonic oscillator problem using operator methods.](http://quantummechanics.ucsd.edu/ph130a/130_notes/node167.html#section:HOop) We write the Hamiltonian in terms of the operator





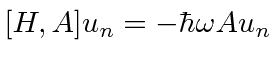
We compute the **commutators**



\begin{displaymath}\bgroup\color{black}[H,A]=\hbar\omega[A^\dagger A,A]=\hbar\omega[A^\dagger,A]A=-\hbar\omega A\egroup\end{displaymath}

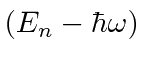
\begin{displaymath}\bgroup\color{black}[H,A^\dagger]=\hbar\omega[A^\dagger A,A^\...
...\hbar\omega A^\dagger[A,A^\dagger]=\hbar\omega A^\dagger\egroup\end{displaymath}

If we apply the the commutator to the eigenfunction \bgroup\color{black}$u_n$\egroup, we get



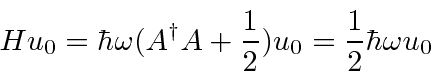
which rearranges to the eigenvalue equation

\begin{displaymath}\bgroup\color{black}H(Au_n)=(E_n-\hbar\omega)(Au_n).\egroup\end{displaymath}

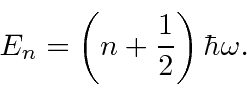
This says that is an eigenfunction of \bgroup\color{black}$H$\egroupwith eigenvalue so it **lowers the energy** by \bgroup\color{black}$\hbar\omega$\egroup. Since the energy must be positive for this Hamiltonian, the lowering must stop somewhere, at the ground state, where we will have

\begin{displaymath}\bgroup\color{black}Au_0=0.\egroup\end{displaymath}

This allows us to compute the **ground state energy** like this



showing that the ground state energy is . Similarly, \bgroup\color{black}$A^\dagger$\egroup**raises the energy** by \bgroup\color{black}$\hbar\omega$\egroup. We can travel up and down the energy ladder using \bgroup\color{black}$A^\dagger$\egroupand \bgroup\color{black}$A$\egroup, always in steps of \bgroup\color{black}$\hbar\omega$\egroup. The energy eigenvalues are therefore



A little more computation shows that

\begin{displaymath}\bgroup\color{black}Au_n=\sqrt{n}u_{n-1}\egroup\end{displaymath}

and that

\begin{displaymath}\bgroup\color{black}A^\dagger u_n=\sqrt{n+1}u_{n+1}.\egroup\end{displaymath}

These formulas are useful for all kinds of **computations** within the important harmonic oscillator system. Both \bgroup\color{black}$p$\egroupand \bgroup\color{black}$x$\egroupcan be written in terms of \bgroup\color{black}$A$\egroupand \bgroup\color{black}$A^\dagger$\egroup.