Chapter 3 Complex numbers and hyperbolic functions

3.1 The need for complex numbers

$$f(z) = z^2 - 4z + 5 = 0$$

$$\Rightarrow z = 2 \pm \frac{\sqrt{-4}}{2} \Rightarrow 2 \pm \sqrt{-1} \quad \text{no real solutions}$$

$$\Rightarrow z = 2 \pm i$$

$$\boxed{\text{define complex number } z = x + iy, \text{ and } i = \sqrt{-1}}$$

$$\begin{cases} \text{the real part x is denoted by Re(z)} \\ \text{the imaginary part y is denoted by Im(z)} \end{cases}$$

$z = x + iy$ can be written $z = (x, y)$

3.2 Manipulation of complex numbers

- addition and subtraction

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

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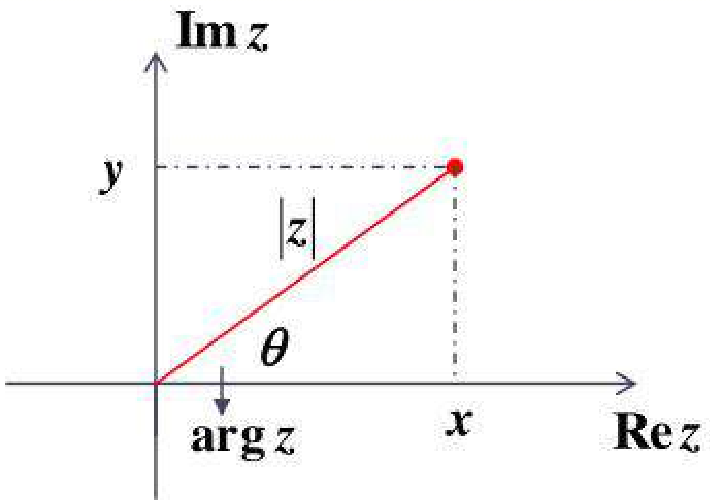
- modulus and argument

$|z| = \sqrt{x^2 + y^2}$ the modulus of z

$\theta = \arg z = \tan^{-1}(y/x)$ the argument of z



Ex: $z = 2 - 3i$

$|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

$\arg z = \tan^{-1}(-3/2)$

- multiplication

$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$

$z_1 z_2 = z_2 z_1$

$(z_1 z_2) z_3 = z_1 (z_2 z_3)$

$|z_1 z_2| = |z_1||z_2|$

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$

- special points $\pm 1, \pm i$

$\arg(1) = 0 \quad \arg(-1) = \pi \quad \arg(i) = \pi/2 \quad \arg(-i) = 3\pi/2$

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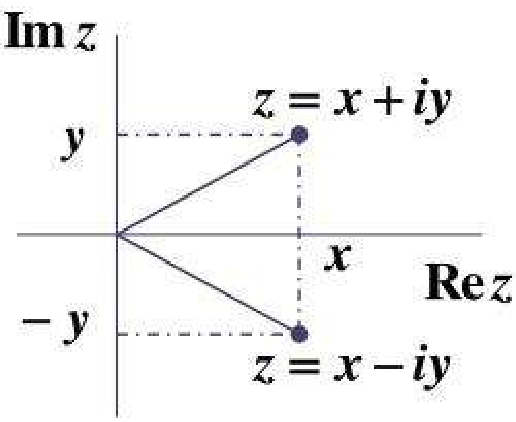
- **complex conjugate**

$$z = x + iy \Rightarrow z^* = x - iy$$

$$zz^* = (x+iy)(x-iy) = x^2 + y^2 = |z|^2$$

z^* **is a reflection of** z **in the real axis**



- **properties of complex conjugate**

$$(z_1 \pm z_2)^* = z_1^* \pm z_2^* \qquad (z^*)^* = z$$

$$(z_1 z_2)^* = z_1^* z_2^* \qquad z + z^* = 2\,\mathrm{Re}\, z = 2x$$

$$(z_1 / z_2)^* = z_1^* / z_2^* \qquad z - z^* = 2i\,\mathrm{Im}\, z = 2iy$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

3.3 Polar representation of complex numbers

- complex exponential function

$$e^{z}=\exp(z)=1+z+\frac{z^{2}}{2!}+\frac{z^{3}}{3!}+\dots\dots\dots.\frac{z^{n}}{n!}$$

$$\Rightarrow e^{z_1}e^{z_2}=e^{z_1+z_2}$$

for $z=i\theta$, θ is real

$$e^{i\theta}=1+i\theta-\frac{\theta^{2}}{2!}-\frac{i\theta^{3}}{3!}+\dots\dots\dots$$

$$=1-\frac{\theta^{2}}{2!}+\frac{\theta^{4}}{4!}+\dots\dots\dots.+i(\theta-\frac{\theta^{3}}{3!}+\frac{\theta^{5}}{5!}+\dots..)$$

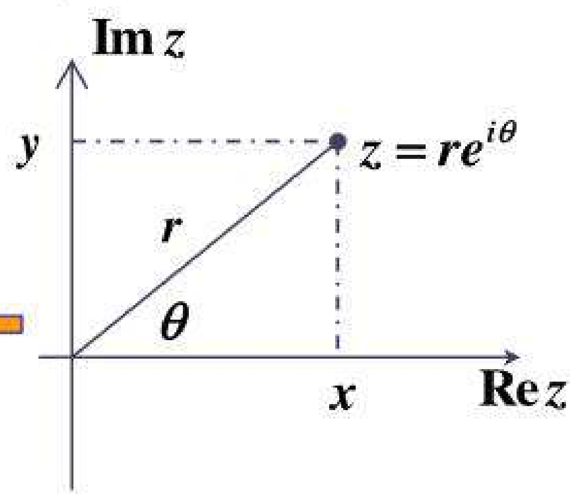
$$=\cos\theta+i\sin\theta$$ **Euler's equation**

$$\Rightarrow e^{in\theta}=\cos(n\theta)+i\sin(n\theta)$$

$$re^{i\theta}=r(\cos\theta+i\sin\theta)=x+iy$$

$$\Rightarrow z=re^{i\theta}\Rightarrow r=|z|\Rightarrow\theta=\arg z$$

$$z=re^{i\theta}=re^{i(\theta+2n\pi)}$$



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- multiplication and division in polar form

$z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$
$$\Rightarrow |z_1 z_2| = |z_1||z_2|$$
$$\Rightarrow \arg(z_1 z_2) = \arg z_1 + \arg z_2$$

← multiplication

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$
$$\Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$
$$\Rightarrow \arg(z_1 / z_2) = \arg z_1 - \arg z_2$$

← division

3.4 de Moivre's theorem

$$(e^{i\theta})^n = e^{in\theta}$$
$$\Rightarrow (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

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Ex. 1: Express $\cos 3\theta$ and $\sin 3\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$

$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\sin\theta\cos^2\theta - \sin^3\theta)$$

$$\boxed{\begin{aligned}&\Rightarrow \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta\\ &\Rightarrow \sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta\end{aligned}}$$

Ex. 2:

$$z^n + \frac{1}{z^n} = e^{in\theta} + e^{-in\theta} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta)$$

$$= 2\cos n\theta$$

$$z^n - \frac{1}{z^n} = 2i\sin n\theta$$

- find the nth roots of unity

$z^n = 1 = e^{i2k\pi}$ k is an integer

$$\Rightarrow z = e^{i2k\pi/n} \Rightarrow z_{1,2,3,\dots,n} = 1, e^{i2\pi/n}, \dots\dots\dots, e^{i2(n-1)\pi/n}$$

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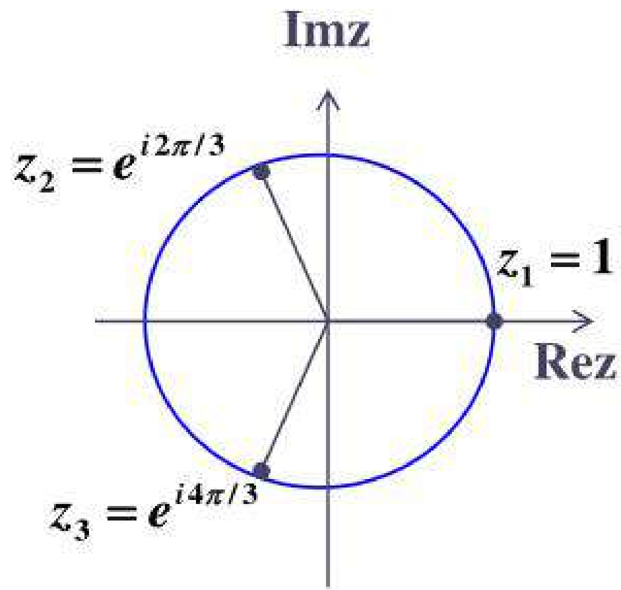
Ex. : Find the solutions to the equation $z^3 = 1$

$z = e^{i2k\pi/3}$

$\Rightarrow z_1 = e^{i0} = 1$

$\Rightarrow z_2 = e^{i2\pi/3} = -1/2 + i\sqrt{3}/2$

$\Rightarrow z_3 = e^{i4\pi/3} = -1/2 - i\sqrt{3}/2$



Ex. : The three roots of $z^3 = 1$ are $1, \ \omega, \ \omega^2$

$$\Rightarrow \omega^3 = 1 \quad and \quad 1 + \omega + \omega^2 = 0$$

Proof:

$$\omega = e^{i2\pi/3}, \quad \omega^2 = e^{i4\pi/3} = e^{-i2\pi/3}$$

$$\Rightarrow 1 + \omega + \omega^2 = 1 + e^{i2\pi/3} + e^{-i2\pi/3} = 1 + 2\cos(2\pi/3) = 0$$

Ex: Solve the polynomial equation

$$f(z)=z^6-z^5+4z^4-6z^3+2z^2-8z+8=0$$

$$try \quad z=1 \Rightarrow f(z=1)=0 \Rightarrow z=1 \quad is \quad a \quad root$$

$$\Rightarrow (z^5+4z^3-2z^2-8)(z-1)=0$$

$$\Rightarrow (z^3-2)(z^2+4)(z-1)=0$$

(1) $z^3=2=2e^{i2k\pi} \quad k=0,1,2,\ldots\ldots$

$$z=2^{1/3}e^{i2k\pi/3}$$

$$\Rightarrow z_1=2^{1/3}$$

$$\Rightarrow z_2=2^{1/3}e^{i2\pi/3}=2^{1/3}(-1/2+i\sqrt{3}/2)$$

$$\Rightarrow z_3=2^{1/3}e^{i4\pi/3}=2^{1/3}e^{-i2\pi/3}=2^{1/3}(-1/2-i\sqrt{3}/2)$$

(2) $z^2+4=0 \Rightarrow z=\pm i\sqrt{4}$

$$\Rightarrow z_4=2i$$

$$\Rightarrow z_5=-2i$$

(3) $z-1=0 \Rightarrow z_6=1$

3.5 Complex logarithms complex powers

$$w = Lnz \Rightarrow z = e^{w}$$

$$\Rightarrow z_1 z_2 = e^{w_1} e^{w_2} = e^{w_1 + w_2}$$

$$\Rightarrow Ln(z_1 z_2) = w_1 + w_2 = Lnz_1 + Lnz_2$$

- **note:**

$$\arg z = \theta + 2n\pi \Rightarrow z = re^{i(\theta + 2n\pi)}$$

$$\Rightarrow Lnz = \ln r + i(\theta + 2n\pi)$$ **multivalued**

$$-\pi < \theta \le \pi \Rightarrow \ln z = Lnz$$ **single valued**

$\ln z$ is the principal value of Lnz

Ex: $Ln(-i) = Ln[e^{i(-\pi/2 + 2n\pi)}] = i(-\pi/2 + 2n\pi)$

$$\Rightarrow \ln(-i) = -i\pi/2$$

Ex: $z = i^{-2i}$

$$Lnz = -2iLni \Rightarrow e^{Lnz} = z = e^{-2iLni}$$

$$Lni = Ln[e^{i(\pi/2 + 2n\pi)}] = i(\pi/2 + 2n\pi)$$

$$\Rightarrow z = i^{-2i} = e^{-2i \cdot i(\pi/2 + 2n\pi)} = e^{\pi + 4n\pi}$$ **is real**

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3.6 Applications to differentiation and integration

Ex: Find the derivative with respect to x of $e^{3x}\cos 4x$

set $z = e^{3x}(\cos 4x + i\sin 4x) = e^{3x}e^{i4x} = e^{(3+4i)x}$

$$dz/dx = (3+4i)e^{(3+4i)x} = (3+4i)e^{3x}(\cos 4x + i\sin 4x)$$

$$\Rightarrow \mathrm{Re}(dz/dx) = (3\cos 4x - 4\sin 4x)e^{3x} = d(e^{3x}\cos 4x)/dx$$

Ex: Evaluate the integral $I = \int e^{ax}\cos bx\,dx$

$$\int e^{ax}(\cos bx + i\sin bx)dx = \int e^{ax}e^{ibx}dx = \int e^{(a+ib)x}dx$$

$$= \frac{e^{(a+ib)x}}{a+ib} + c = \frac{(a-ib)e^{(a+ib)x}}{a^2+b^2} + c$$

$$= \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx + c_1 + i(a\sin bx - b\cos bx) + c_2]$$

$$\Rightarrow \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c_1$$

$$\Rightarrow \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c_2$$

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3.7 Hyperbolic functions

- **definition:**

$$\cosh x = \frac{1}{2}(e^{x} + e^{-x})$$

$$\mathrm{sech}\,x = \frac{1}{\cosh x} = \frac{2}{e^{x} + e^{-x}}$$

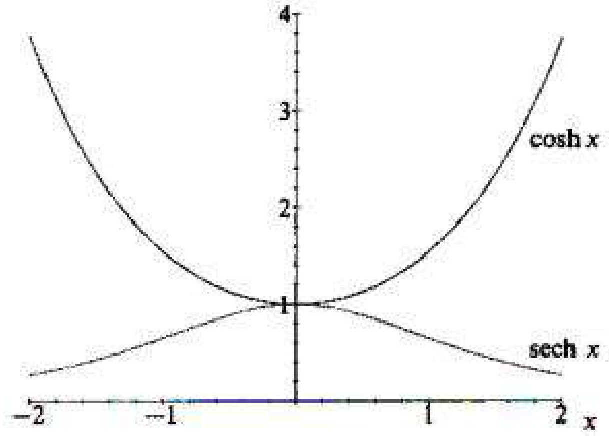

Figure 3.11 Graphs of cosh x and sechx.

$$\sinh x = \frac{1}{2}(e^{x} - e^{-x})$$

$$\mathrm{cosech}\,x = \frac{1}{\sinh x} = \frac{2}{e^{x} - e^{-x}}$$

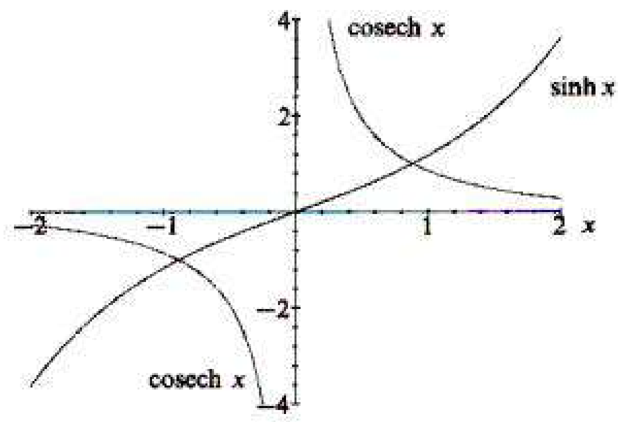

Figure 3.12 Graphs of sinh x and cosechx.

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$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

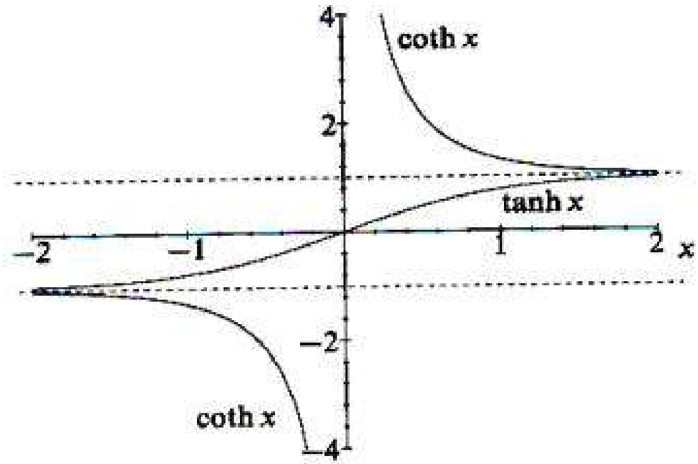

Figure 3.13 Graphs of tanh x and coth x.

- **hyperbolic-trigonometric analogies**

$$\boxed{u\sin g \quad \cos z = (e^{iz} + e^{-iz})/2 \quad \sin z = (e^{iz} - e^{-iz})/2 \quad for \quad z = ix}$$

$$\cos ix = (e^x + e^{-x})/2 = \cosh x$$

$$\sin ix = i(e^x - e^{-x})/2 = i \sinh x$$

$$\cosh x = (e^{ix} + e^{-ix})/2 = \cos x$$

$$\sinh x = (e^{ix} - e^{-ix})/2 = i \sin x$$

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- identities of hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$
$$\mathrm{sec}h^2 x = 1 - \tanh^2 x$$
$$\mathrm{cos}ech^2 x = \coth^2 x - 1$$
$$\sinh 2x = 2\sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

- solving hyperbolic equations

Ex: Solve the hyperbolic equation $\cosh x - 5\sinh x - 5 = 0$

$$\Rightarrow (e^x + e^{-x})/2 - 5(e^x - e^{-x})/2 - 5 = 0$$
$$\Rightarrow -2e^x + 3e^{-x} - 5 = 0$$
$$y = e^x \Rightarrow e^{-x} = 1/y \Rightarrow -2y + 3/y - 5 = 0$$
$$\Rightarrow 2y^2 + 5y - 3 = 0 \Rightarrow (2y-1)(y+3) = 0$$

(1) $2y - 1 = 0 \Rightarrow y = 1/2 = e^x \Rightarrow x = \ln(1/2) = -\ln 2$

(2) $y + 3 = 0 \Rightarrow y = -3 = e^x$

$$\Rightarrow x = \ln(-3) = \ln(i^2 \times 3) = \ln 3 + 2\ln e^{i\pi/2} = \ln 3 + i\pi$$

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- **inverse of hyperbolic functions**

Ex: Find a closed-form expression for the inverse hyperbolic function (1) $y=\cosh^{-1}x$

(2) $y=\sinh^{-1}x$ (3) $y=\tanh^{-1}x$

(1)

$$y=\cosh^{-1}x \Rightarrow x=\cosh y$$

$$\Rightarrow e^{y}=\cosh y+\sinh y=\cosh y+\sqrt{\cosh^2 y-1}$$

$$=x+\sqrt{x^2-1}$$

$$\Rightarrow \boxed{y=\cosh^{-1}x=\ln(\sqrt{x^2-1}+x)}$$

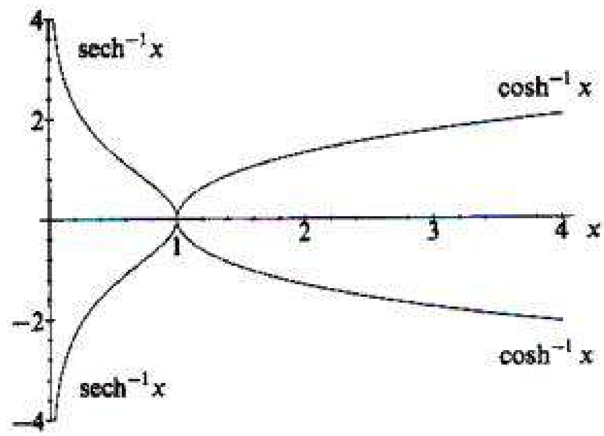

Figure 3.14 Graphs of $\cosh^{-1}x$ and $\operatorname{sech}^{-1}x$.

(2)

$$y=\sinh^{-1}x \Rightarrow x=\sinh y$$

$$\Rightarrow e^{y}=\cosh y+\sinh y=\sqrt{1+\sinh^{2}y}+\sinh y$$

$$=\sqrt{x^{2}+1}+x$$

$$\Rightarrow \boxed{y=\sinh^{-1}x=\ln(\sqrt{1+x^{2}}+x)}$$

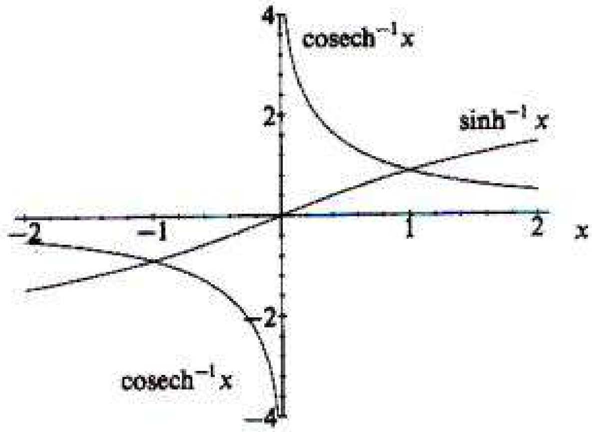

Figure 3.15 Graphs of $\sinh^{-1}x$ and $\text{cosech}^{-1}x$.

(3)

$$y=\tanh^{-1}x \Rightarrow x=\tanh y$$

$$\Rightarrow x=\frac{e^{y}-e^{-y}}{e^{y}+e^{-y}} \Rightarrow (x+1)e^{-y}=(1-x)e^{y}$$

$$\Rightarrow e^{2y}=\frac{1+x}{1-x} \Rightarrow e^{y}=\sqrt{\frac{1+x}{1-x}}$$

$$\Rightarrow \boxed{\ln e^{y}=y=\tanh^{-1}x=\ln(\sqrt{\frac{1+x}{1-x}})=\frac{1}{2}\ln(\frac{1+x}{1-x})}$$

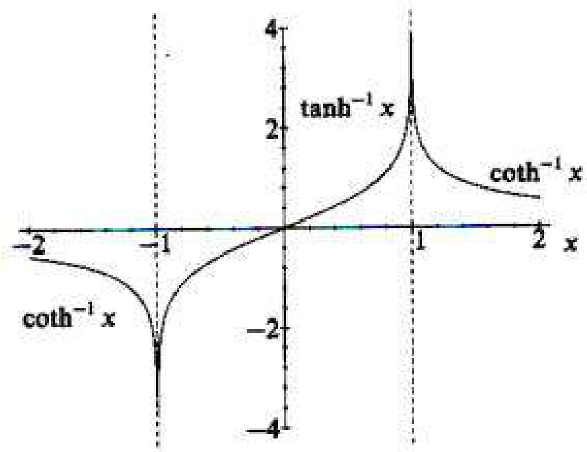

Figure 3.16 Graphs of $\tanh^{-1}x$ and $\coth^{-1}x$.

Chapter 3 Complex numbers and hyperbolic functions

- calculus of hyperbolic functions

$$\frac{d}{dx}\cosh x = \sinh x \qquad \frac{d}{dx}\mathrm{sech}x = -\mathrm{sech}x\tanh x$$

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\mathrm{cosech}x = -\mathrm{cosech}x\coth x$$

$$\frac{d}{dx}\tanh x = \mathrm{sech}^2 x \qquad \frac{d}{dx}\coth x = -\mathrm{cosech}^2 x$$

$$\frac{d}{dx}(\cosh^{-1}\frac{x}{a}) = \frac{1}{\sqrt{x^2-a^2}} \quad for \quad x^2 > a^2$$

$$\frac{d}{dx}(\sinh^{-1}\frac{x}{a}) = \frac{1}{\sqrt{x^2+a^2}}$$

$$\frac{d}{dx}(\tanh^{-1}\frac{x}{a}) = \frac{a}{a^2-x^2} \quad for \quad x^2 < a^2$$

$$\frac{d}{dx}(\coth^{-1}\frac{x}{a}) = \frac{-a}{x^2-a^2} \quad for \quad x^2 > a^2$$