Part 1 Introduction to Financial Management (Chapters 1, 2, 3, 4)

Part 2 Valuation of Financial Assets (Chapters 5, 6, 7, 8, 9, 10)

Part 3 Capital Budgeting (Chapters 11, 12, 13, 14)

Part 4 Capital Structure and Dividend Policy (Chapters 15, 16)

Part 5 Liquidity Management and Special Topics in Finance (Chapters 17, 18, 19, 20)

CHAPTER 5

The Time Value of Money
The Basics

Chapter **Outline**

5.1 Using Timelines to Visualize Cash Flows **(pgs. 162–163)** → **Objective 1.** Construct cash flow timelines to organize your analysis of problems involving the time value of money.

5.2 Compounding and Future Value **(pgs. 164–171)** → **Objective 2.** Understand compounding and calculate the future value of cash flows using mathematical formulas, a financial calculator, and an Excel spreadsheet.

5.3 Discounting and Present Value **(pgs. 171–176)** → **Objective 3.** Understand discounting and calculate the present value of cash flows using mathematical formulas, a financial calculator, and an Excel spreadsheet.

5.4 Making Interest Rates Comparable **(pgs. 177–181)** → **Objective 4.** Understand how interest rates are quoted and know how to make them comparable.

Principle P1 Applied

Chapters 5 and 6 are dedicated to P Principle 1: **Money Has a Time Value**. This basic idea—a dollar received today, other things being the same, is worth more than a dollar received a year from now—underlies many financial decisions faced in business. In Chapter 1, we discussed capital budgeting, capital structure, and working capital management decisions—each of these decisions involves aspects of the time value of money. In this chapter, we learn how to calculate the value today of money you will receive in the future as well as the future value of money you have today. In Chapter 6, we extend our analysis to multiple cash flows spread out over time. In later chapters, we will use the skills we gain from Chapters 5 and 6 to analyze bond prices (Chapter 9) and stock prices (Chapter 10), calculate the value of investment opportunities (Chapters 11–13), and determine the cost of financing a firm's investments (Chapter 14).

Payday Loans

Sometimes marketed to college students as quick relief for urgent expenses, a payday loan is a short-term loan to cover expenses until the next payday. As some borrowers turn to these loans during times of financial desperation, lenders can charge them extremely high rates of interest. For example, in 2016, one payday lender, Advance America,[1] advertised that you could borrow $100 and repay $126.40 in 14 days. This might not sound like a bad deal on the surface, but if we apply some basic rules of finance to analyze this loan, we see quite a different story. The effective annual interest rate for this payday loan is a whopping 44,844 percent! (We will examine this later in the chapter in section 5.4, pages 177–181.)

The very high rates of interest charged by these lenders have led some states to impose limits on the interest rates payday lenders can charge. Even so, the cost of this type of loan can be extremely high. Understanding the time value of money is an essential tool to analyzing the cost of this and other types of financing.

[1]https://www.advanceamerica.net/apply-for-a-loan/service/payday-loan accessed June 28, 2016.

Regardless of Your

"A Dollar Saved Is Two Dollars Earned"

Suppose that you and your classmate each receive a gift of $10,000 from grandparents but choose different ways to invest the new-found money. You immediately invest your gift until retirement, whereas your classmate carries around his gift in his wallet in the form of 100 crisp $100 bills. Then, after 15 years of carrying around a fat wallet, your classmate decides to invest his $10,000 for retirement.

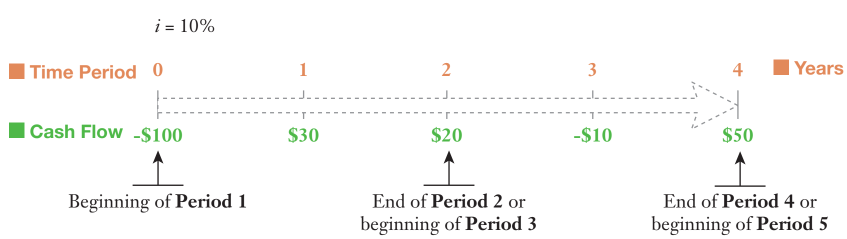
If you invest your $10,000 for 46 years and earn 10 percent per year until you retire, you'll end up with over $800,000. If your classmate invests his $10,000 for 31 years (remember that he carried his money around for 15 years in his wallet) and earns the same 10 percent per year, he'll end up with only about $192,000. Knowing about the power of the time value of money provided you with an additional $600,000 at retirement. In this chapter, we'll learn more about these kinds of valuation problems. For now, keep in mind that the time value of money is a concept you will want to understand, regardless of your major.

Your Turn: See Study Questions 5–7 and 5–8.

5.1 Using Timelines to Visualize Cash Flows

To evaluate a new project, a financial manager must be able to compare benefits and costs that occur at different times. We will use the time-value-of-money tools we develop in this chapter to make the benefits and costs comparable, allowing us to make logical decisions. We begin our study of time value analysis by introducing some basic tools. As a first step, we can construct a **timeline**, a linear representation of the timing of cash flows. A timeline identifies the timing and amount of a stream of payments—both cash received and cash spent—along with the interest rate earned. Timelines are a critical first step that financial analysts use to solve financial problems, and we will refer to timelines throughout this text.

To learn how to construct a timeline, consider an example where we have annual cash inflows and outflows over the course of four years. The following timeline illustrates these cash inflows and outflows from Time Period or Year 0 (the present) until the end of Year 4:



For our purposes, time periods are identified above the timeline. In this example, the time periods are measured in years, indicated on the far right of the timeline. For example, Time Period 0 in this example is the current year. The dollar amount of the cash flow received or spent during each time period is shown below the timeline. Positive values represent *cash inflows*. Negative values represent *cash outflows*. For example, in the timeline shown, a $100 cash outflow (a negative cash flow) occurs at the beginning of the first year (at Time Period 0), followed by cash inflows (positive cash flows) of $30 and $20 in Years 1 and 2, a cash outflow of $10 in Year 3, and, finally, a cash inflow of $50 in Year 4.

Timelines are typically expressed in years, but they could be expressed in months, days, or, for that matter, any unit of time. For now, let's assume we're looking at cash flows that occur annually, so the distance between 0 and 1 represents the time period between today and the end of the first year. The interest rate—10 percent, in this example—is listed above the timeline.

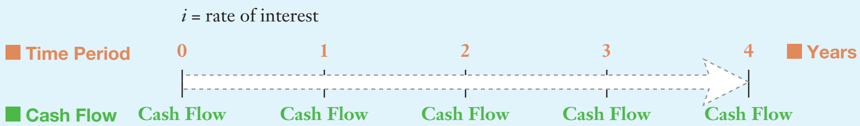
Checkpoint 5.1 MyLab Finance Video

Creating a Timeline

Suppose you lend a friend $10,000 today to help him finance a new Jimmy John's Sandwiches franchise and in return he promises to give you $12,155 at the end of the fourth year. How can one represent this as a timeline? Note that the interest rate is 5 percent.

STEP 1: Picture the problem

A timeline provides a tool for visualizing cash flows and time:



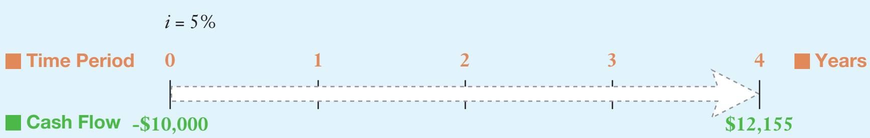
STEP 2: Decide on a solution strategy

To complete the timeline, we simply record the cash flows on the template.

STEP 3: Solve

We can input the cash flows for this investment on the timeline as shown below. Time Period 0 (the present) is shown at the left end of the timeline, and future time periods are shown above the timeline, moving from left to right; each cash flow is listed below the timeline at the appropriate time period.

Keep in mind that Year 1 represents the end of the first year as well as the beginning of the second year.



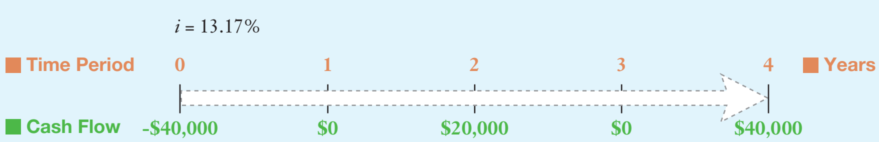
STEP 4: Analyze

Using timelines to visualize cash flows is useful in financial problem solving. From analyzing the timeline, we can see that there are two cash flows, an initial $10,000 cash outflow and a $12,155 cash inflow at the end of Year 4.

STEP 5: Check yourself

Draw a timeline for an investment of $40,000 today that returns nothing in Year 1, $20,000 at the end of Year 2, nothing in Year 3, and $40,000 at the end of Year 4; the interest rate is 13.17 percent.

ANSWER:



>> END Checkpoint 5.1

Before you move on to 5.2

Concept Check | 5.1

1. What is a timeline, and how does it help you solve problems involving the time value of money?
2. Does Year 5 represent the end of the fifth year, the beginning of the sixth year, or both?

5.2 Compounding and Future Value

If we assume that an investment will earn interest only on the original principal, we call this **simple interest**. Suppose that you put $100 in a savings account earning 6 percent interest annually. How much will your savings grow after one year? If you invest for one year at an interest rate of 6 percent, you will earn 6 percent simple interest on your initial deposit of $100, giving you a total of $106 in your account. What if you leave your $100 in the bank for two years? In this case, you will earn interest not only on your original $100 deposit but also on the $6 in interest you earned during the first year. This process of accumulating interest on an investment over multiple time periods is called **compounding**. And when interest is earned on both the initial principal and the reinvested interest during prior periods, the result is called **compound interest**.

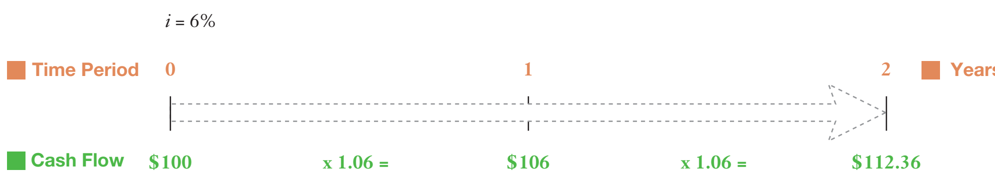
Time-value-of-money calculations are essentially comparisons between what we will refer to as **present value**, what a cash flow is worth to you today, and **future value**, what a cash flow will be worth to you in the future. The following is a mathematical formula that shows how these concepts relate to each other when the future value is in one year:

$$\text{Future Value in 1 Year} = \text{Present Value} \times (1 + \text{Interest Rate}) \qquad \textbf{(5–1)}$$

In the savings account example, you began with a $100 investment, so the present value is $100. The future value in one year is then given by the equation

$$\$100 \times (1 + .06) = \$106.00$$

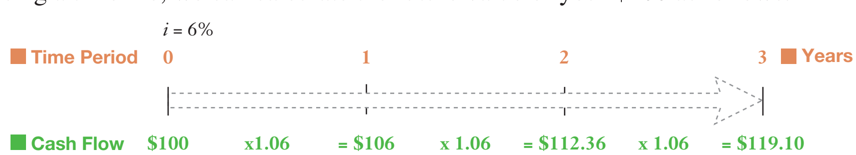
To see how to calculate the future value in two years, let's do a timeline and a few calculations:



During the first year, your $100 deposit earns $6 in interest. Summing the interest and the original deposit gives you a balance of $106 at the end of the first year. In the second year, you earn $6.36 in interest, giving you a future value of $112.36. Why do you earn $0.36 more in interest during the second year than during the first? Because in the second year, you earn an additional 6 percent on the $6 in interest you earned in the first year. This amounts to $0.36 (or $6 × .06). Again, this result is an example of compound interest. Anyone who has ever had a savings account or purchased a government savings bond has received compound interest.

What happens to the value of your investment at the end of the third year, assuming the same interest rate of 6 percent? We can follow the same approach to calculate the future value in three years.

Using a timeline, we can calculate the future value of your $100 as follows:



Note that every time we extend the analysis for one more period, we just multiply the previous balance by (1 + Interest Rate). Consequently, we can use the following equation to express the future value of any amount of money for any number of periods (where n = the number of periods during which the compounding occurs):

$$\begin{matrix}\text{Future}\\ \text{Value}_{\text{Period } n}\end{matrix} = \begin{matrix}\text{Present}\\ \text{Value (Deposit)}\end{matrix}\left(1 + \begin{matrix}\text{Interest}\\ \text{Rate } (i)\end{matrix}\right)^{n}$$

or

$$\begin{matrix}\text{Future Value}\\ \text{in Year } n\\ (FV_n)\end{matrix} = \begin{matrix}\text{Present}\\ \text{Value } (PV)\end{matrix}\left(1 + \begin{matrix}\text{Annual}\\ \text{Interest Rate } (i)\end{matrix}\right)^{\text{Number of Years } (n)} \qquad \textbf{(5–1a)}$$

Important Definitions and Concepts:

- FV_n = the future value of the investment at the end of n periods.
- i = the interest (or growth) rate per period.
- PV = the present value, or original amount invested at the beginning of the first period.

We also refer to $(1 + i)^n$ as the **future value interest factor**. To find the future value of a dollar amount, simply multiply that dollar amount by the appropriate future value interest factor:

$$\begin{aligned} FV_n &= PV(1 + i)^n \\ &= PV \times \text{ Future Value Interest Factor} \end{aligned}$$

where Future Value Interest Factor $= (1 + i)^n$.

Panel A in Figure 5.1 shows what your investment of $100 will grow to in four years if it continues to earn an annual compound interest rate of 6 percent. Notice how the amount of interest earned increases each year. In the first year, you earn only $6 in interest, but by Year 4, you earn $7.15.

Prior to the introduction of inexpensive financial calculators and Excel, future values were commonly calculated using time-value-of-money tables containing future value interest factors for different combinations of i and n. Table 5.1 provides an abbreviated future value interest factor table; you can find the expanded future value interest factor tables in Appendix B in MyLab Finance. So to find the value of $100 invested for four years at 6 percent, we would simply look at the intersection of the $n = 4$ row and the 6% column, which is the future value interest factor of 1.262. We would then multiply this value by $100 to find that our investment of $100 at 6 percent for four years would grow to $126.20.

Compound Interest and Time

As Panel B of Figure 5.1 shows, the future value of an investment grows with the number of periods we let it compound. For example, after five years, the future value of $100 earning 10 percent interest each year will be $161.05. However, after 25 years, the future value of that investment will be $1,083.47. Note that although we increased the number of years threefold, the future value increases more than sixfold ($1,083.47/$161.05 = 6.7-fold). This illustrates an important point: Future value is not directly proportional to time. Instead, future value grows exponentially. This means it grows by a fixed percentage each year, which means that the dollar value grows by an increasing amount each year.

Compound Interest and the Interest Rate

Panel C of Figure 5.1 illustrates that future value increases dramatically with the level of the rate of interest. For example, the future value of $100 in 25 years, given a 10 percent interest rate compounded annually, is $1,083.47. However, if we double the rate of interest to 20 percent, the future value increases almost ninefold in 25 years to $9,539.62. This illustrates another important point: The increase in future value is not directly proportional to the increase in the rate of interest. We doubled the rate of interest, and the future value of the investment increased by 8.8 times. Why did the future value jump by so much? Because there is a lot of time over 25 years for the higher interest rate to result in more interest being earned on interest.

Techniques for Moving Money Through Time

In this book, we will refer to three methods for solving problems involving the time value of money: mathematical formulas, financial calculators, and spreadsheets.

- **Do the math.** You can use the mathematical formulas just as we have done in this chapter. You simply substitute the values that you know into the appropriate time-value-of-money equation to find the answer.
- **Use a financial calculator.** Financial calculators have preprogrammed functions that make time-value-of-money calculations simple.

Figure 5.1

Future Value and Compound Interest Illustrated
(Panel A) Calculating Compound Interest

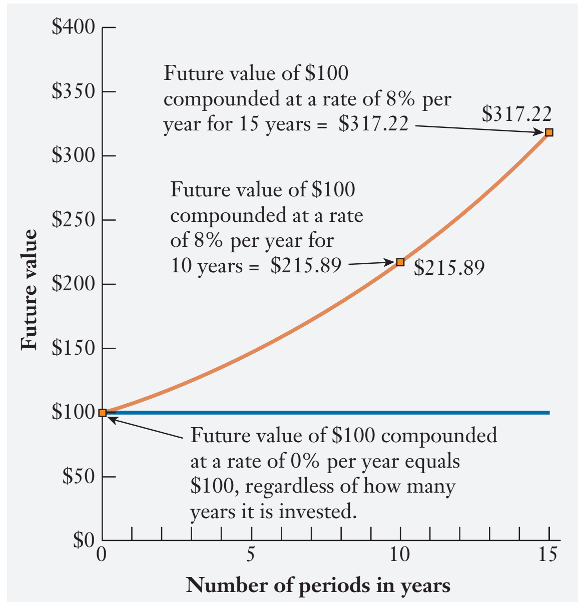
This panel shows how interest compounds annually. During the first year, $100 invested at a 6% interest rate earns only $6. Because we earn 6% on the ending value for Year 1 (or $106), we earn $6.36 in interest in Year 2. This increase in the amount of interest results from interest being earned on both the initial deposit of $100 and the $6.00 in interest earned during Year 1. The fact that we earn interest on both principal and interest is why we refer to this as compound interest. Simple interest, on the other hand, would be earning only $6.00 in interest each and every year.

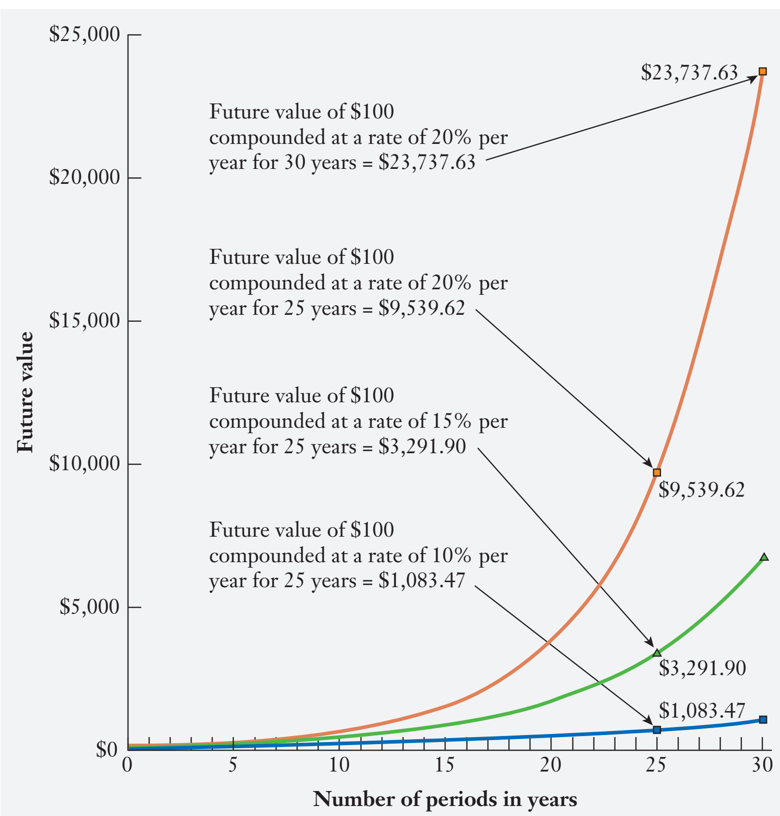
Interest Earned = Beginning Value × Interest Rate

Year	Beginning Value	Interest Earned	Ending Value
1	$ 100.00	**$ 6.00**	$ 106.00
2	$ 106.00	**$ 6.36**	$ 112.36
3	$ 112.36	**$ 6.74**	$ 119.10
4	$ 119.10	**$ 7.15**	$ 126.25

(Panel B) The Power of Time

This figure illustrates the importance of time when it comes to compounding. Because interest is earned on past interest, the future value of $100 deposited in an account that earns 8% interest compounded annually grows over threefold in 15 years. If we were to expand this figure to 45 years (which is about how long you have until you retire, assuming you're around 20 years old right now), the account would grow to over 31 times its initial value.





(Panel C) The Power of the Rate of Interest

This figure illustrates the importance of the interest rate in the power of compounding. As the interest rate climbs, so does the future value. In fact, when we change the interest rate from 10% to 20%, the future value in 25 years increases by 8.8 times, jumping from $1,083.47 to $9,539.62.

>> END FIGURE 5.1

Table 5.1 Future Value Interest Factors

Number of Periods (n)	$i = 3\%$	$i = 6\%$	$i = 9\%$	$i = 12\%$
1	1.030	1.060	1.090	1.120
2	1.061	1.124	1.188	1.254
3	1.093	1.191	1.295	1.405
4	1.126	1.262	1.412	1.574

- **Use a spreadsheet on your personal computer.** Spreadsheet software such as Excel has preprogrammed functions built into it. The same inputs that are used with a financial calculator are also used as inputs to Excel. As a result, if you can correctly set a problem up to solve on your financial calculator, you can easily set it up to solve using Excel. In the business world, Excel is the spreadsheet of choice and is the most common way of moving money through time.

In Appendix A in MyLab Finance, we show you how to solve valuation problems using each of these methods. Because we, the authors of this book, believe that spending enough time solving problems the old-fashioned way—by doing the math—leads to a deeper understanding and better retention of the concepts found in this book, we will first demonstrate how to solve problems using the formulas. However, we will also demonstrate, whenever possible, how to derive solutions using a financial calculator and Excel.

Applying Compounding to Things Other Than Money

Although this chapter focuses on moving money through time at a given interest rate, the concept of compounding applies to almost anything that grows. For example, let's suppose we're interested in knowing how big the market for wireless printers will be in five years and we assume the demand for them will grow at a rate of 25 percent per year over those five years. We can calculate the future value of the market for printers using the same formula we used to calculate the future value for a sum of money. If the market is currently 25,000 printers per year, then 25,000 would be PV, n would be 5, and i would be 25 percent. Substituting into Equation (5–1a), we would solve for FV:

$$\begin{matrix}\text{Future Value}\\ \text{in Year } n\\ (FV_n)\end{matrix} = \begin{matrix}\text{Present}\\ \text{Value } (PV)\end{matrix}\left(1 + \begin{matrix}\text{Annual}\\ \text{Interest Rate } (i)\end{matrix}\right)^{\substack{\text{Number of}\\ \text{Years } (n)}} = 25{,}000\,(1 + .25)^5 = 76{,}293$$

The power of compounding can also be illustrated through the story of a peasant who wins a chess tournament sponsored by the king. The king then asks him what he would like as his prize. The peasant answers that, for his village, he would like one grain of wheat to be placed on the first square of his chessboard, two pieces on the second square, four on the third square, eight on the fourth square, and so forth until the board is filled up. The king, thinking he was getting off easy, pledged his word of honor that this would be done. Unfortunately for the king, by the time all 64 squares on the chessboard were filled, there were 18.5 million trillion grains of wheat on the board because the kernels were compounding at a rate of 100 percent over the 64 squares. In fact, if the kernels were one-quarter inch long, they would have stretched, if laid end to end, to the sun and back 391,320 times! Needless to say, no one in the village ever went hungry. What can we conclude from this story? There is incredible power in compounding.

Compound Interest with Shorter Compounding Periods

So far we have assumed that the compounding period is always a year in length. However, this isn't always the case. For example, banks often offer savings accounts that compound interest every day, month, or quarter. Savers prefer more frequent compounding because they earn interest on their interest sooner and more frequently. Fortunately, it's easy to adjust for different compounding periods, and later in the chapter, we will provide more details on how to compare two loans with different compounding periods.

Checkpoint 5.2

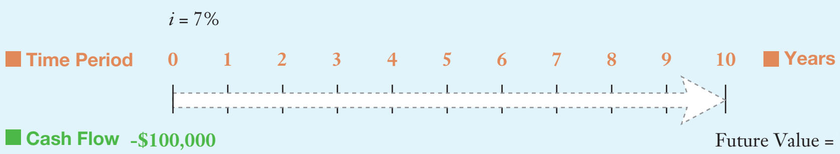
MyLab Finance Video

Calculating the Future Value of a Cash Flow

You are put in charge of managing your firm's working capital. Your firm has \$100,000 in extra cash on hand and decides to put it in a savings account paying 7 percent interest compounded annually. How much will your firm have in its savings account in 10 years?

STEP 1: Picture the problem

We can set up a timeline to identify the cash flows from the investment as follows:



STEP 2: Decide on a solution strategy

This is a simple future value problem. We can find the future value using Equation (5–1a).

STEP 3: Solve

Using the Mathematical Formulas. Substituting PV = \$100,000, i = 7%, and n = 10 years into Equation (5–1a), we get

$$\text{Future Value in Year } n\ (FV_n) = \text{Present Value } (PV)\left(1 + \text{Annual Interest Rate } (i)\right)^{\text{Number of Years } (n)} \qquad \textbf{(5–1a)}$$

$$\begin{aligned} FV_n &= \$100{,}000(1 + .07)^{10} \\ &= \$100{,}000\ (1.96715) \\ &= \$196{,}715 \end{aligned}$$

At the end of 10 years, the firm will have \$196,715 in its savings account.

Using a Financial Calculator.

Enter	10	7.0	–100,000	0	
	N	I/Y	PV	PMT	FV
Solve for					196,715

Using an Excel Spreadsheet.

= FV(rate,nper,pmt,pv) or, with values entered, = FV(0.07,10,0,–100000)

STEP 4: Analyze

Notice that you input the present value with a negative sign because present value represents a cash outflow. In effect, the money leaves your firm when it's first invested. In this problem, your firm invested \$100,000 at 7 percent and found that it will grow to \$196,715 after 10 years. Put another way, given a 7 percent compound rate, your \$100,000 today will be worth \$196,715 in 10 years.

STEP 5: Check yourself

What is the future value of \$10,000 compounded at 12 percent annually for 20 years?

ANSWER: \$96,462.93.

Your Turn: For more practice, do related Study Problems 5–1, 5–2, 5–4, 5–6, and 5–8 through 5–11 at the end of this chapter.

>> END Checkpoint 5.2

Consider the following example: You invest \$100 for five years at an interest rate of 8 percent, and the investment is compounded semiannually (twice a year). This means that interest is calculated every six months. Essentially, you are investing your money for 10 six-month periods, and in each period, you will receive 4 percent interest. In effect, we divide the annual interest rate (i) by the number of compounding periods per year (m), and we multiply the number of years (n) times the number of compounding periods per year (m) to convert the number of years into the number of periods. So our future value formula found in Equation (5 –1a) must be adjusted as follows:

$$\begin{array}{c}\text{Future Value}\\ \text{in Year } n\\ (FV_n)\end{array} = \begin{array}{c}\text{Present}\\ \text{Value } (PV)\end{array}\left(1+\frac{\begin{array}{c}\text{Annual}\\ \text{Interest Rate } (i)\end{array}}{\begin{array}{c}\text{Compounding}\\ \text{Periods per Year } (m)\end{array}}\right)^{m\,\times\,(\text{Number of Years } (n))} \tag{5–1b}$$

Substituting into Equation (5–1b) gives us the following estimate of the future value in five years:

$$\begin{aligned} FV_n &= \$100(1 + .08/2)^{2\times 5}\\ &= \$100(1.4802)\\ &= \$148.02 \end{aligned}$$

If the compounding had been annual rather than semiannual, the future value of the investment would have been only \$146.93. Although the difference here seems modest, it can be significant when large sums of money are involved and the number of years and the number of compounding periods within those years are both large. For example, for your \$100 investment, the difference is only \$1.09. But if the amount was \$50 million (not an unusually large bank balance for a major company), the difference would be \$545,810.41.

Table 5.2 shows how shorter compounding periods lead to higher future values. For example, if you invested \$100 at 15 percent for one year and the investment was compounded daily rather than annually, you would end up with \$1.18 (\$116.18 − \$115.00) more. However, if the period was extended to 10 years, then the difference would grow to \$43.47 (\$448.03 − \$404.56).

Table 5.2 The Value of \$100 Compounded at Various Non-annual Periods and Various Rates

Notice that the impact of shorter compounding periods is heightened by both higher interest rates and compounding over longer time periods.

For 1 Year at *i* Percent	***i* = 2%**	**5%**	**10%**	**15%**	
Compounded annually	\$102.00	\$105.00	\$110.00	**\$115.00**	
Compounded semiannually	102.01	105.06	110.25	115.56	
Compounded quarterly	102.02	105.09	110.38	115.87	\$1.18
Compounded monthly	102.02	105.12	110.47	116.08	
Compounded weekly (52)	102.02	105.12	110.51	116.16	
Compounded daily (365)	102.02	105.13	110.52	**116.18**	
For 10 Years at i Percent	***i* = 2%**	**5%**	**10%**	**15%**	
Compounded annually	\$121.90	\$162.89	\$259.37	**\$404.56**	
Compounded semiannually	122.02	163.86	265.33	424.79	
Compounded quarterly	122.08	164.36	268.51	436.04	\$43.47
Compounded monthly	122.12	164.70	270.70	444.02	
Compounded weekly (52)	122.14	164.83	271.57	447.20	
Compounded daily (365)	122.14	164.87	271.79	**448.03**	

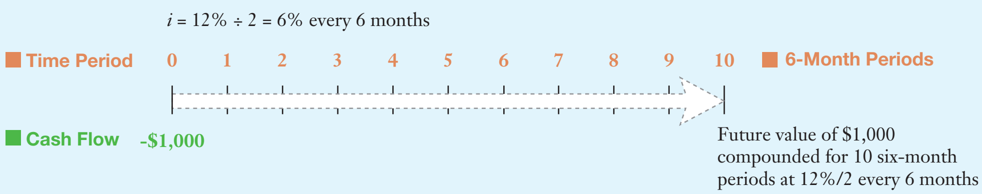
Checkpoint 5.3

Calculating Future Values Using Non-annual Compounding Periods

You have been put in charge of managing your firm's cash position and have noticed that the Plaza National Bank of Portland, Oregon, has recently decided to begin paying interest compounded semiannually instead of annually. If you deposit $1,000 with Plaza National Bank at an interest rate of 12 percent, what will your firm's account balance be in five years?

STEP 1: Picture the problem

If you earn a 12 percent annual interest rate compounded semiannually for five years, you really earn 6 percent every six months for 10 six-month periods. Expressed as a timeline, this problem would look like the following:



STEP 2: Decide on a solution strategy

In this instance, we are simply solving for the future value of $1,000. The only twist is that interest is calculated on a semiannual basis. Thus, if you earn 12 percent interest compounded semiannually for five years, you really earn 6 percent every six months for 10 six-month periods. We can calculate the future value of the $1,000 investment using Equation (5–1b).

STEP 3: Solve

Using the Mathematical Formulas. Substituting number of years (n) = 5, number of compounding periods per year (m) = 2, annual interest rate (i) = 12%, and PV = $1,000 into Equation (5–1b):

$$\begin{array}{c}\text{Future Value}\\ \text{in Year } n\\ (FV_n)\end{array} = \begin{array}{c}\text{Present}\\ \text{Value }(PV)\end{array}\left(1 + \frac{\begin{array}{c}\text{Annual}\\ \text{Interest Rate}(i)\end{array}}{\begin{array}{c}\text{Compounding}\\ \text{Periods per Year }(m)\end{array}}\right)^{m \times (\text{Number of Years }(n))}$$

$$FV_n = \$1{,}000\left(1 + \frac{.12}{2}\right)^{2\times 5} = \$1{,}000 \times 1.79085 = \$1{,}790.85$$

Using a Financial Calculator.

Enter	10	6.0	-1,000	0	
	N	I/Y	PV	PMT	FV
Solve for					1,790.85

You will have $1,790.85 at the end of five years.

Using an Excel Spreadsheet.
= FV(rate,nper,pmt,pv) or, with values entered, = FV(0.06,10,0,–1000)

STEP 4: Analyze

The more often interest is compounded per year—that is, the larger m is—the larger the future value will be. That's because you are earning interest more often on the interest you've previously earned.

STEP 5: Check yourself

If you deposit $50,000 in an account that pays an annual interest rate of 10 percent compounded monthly, what will your account balance be in 10 years?

ANSWER: $135,352.07.

Your Turn: For more practice, do related **Study Problems** 5–5 and 5–7 at the end of this chapter. **>> END Checkpoint 5.3**

Finance for Life

Saving for Your First House

There was a time in the early and mid-2000s when you didn't need to worry about a down payment when you bought a new house. But that all changed as the housing bubble burst and home prices fell. Today, you may be able to get away with putting down only around 10 percent, but the rate on your mortgage will be lower if you can come up with 20 percent. To buy a median-priced home, which was just over $180,000 at the beginning of 2016 (new houses were considerably more than that), you'd have to come up with a 10 percent down payment of $18,000 or a 20 percent down payment of $36,000. On top of that, you would need to furnish your new home, and that costs money, too.

Putting into practice what you have learned in this chapter, you know that the sooner you start to save for your first home, the easier it will be. Once you estimate how much you'll need for that new house, you can easily calculate how much you'll need to save annually to reach your goal. All you need to do is look at two variables: *n* (the number of years you'll be saving the money) and *i* (the interest rate at which your savings will grow). You can start saving earlier, which gives you a larger value for *n*. Or you can earn more on your investments—that is, invest at a higher value for *i*. Of course, you always prefer getting a higher *i* on your savings, but this is not something you can control.

First, let's take a look at a higher value for *i*, which translates into a higher return. For example, let's say you've just inherited $10,000 and you invest it at 6 percent annually for 10 years—after which you want to buy your first house. The calculation is easy. At the end of 10 years, you will have accumulated $17,908 on this investment. But suppose you are able to earn 12 percent annually for 10 years. What will the value of your investment be then? In this case, your investment will be worth $31,058. Needless to say, the rate of interest that you earn plays a major role in determining how quickly your investment will grow.

Now consider what happens if you wait five years before investing your $10,000. The value of *n* drops from 10 to 5, and, as a result, the amount you save also drops. In fact, if you invest your $10,000 for five years at 6 percent, you end up with $13,382, and even at 12 percent, you end up with only $17,623.

The bottom line is this: The earlier you begin saving, the more impact every dollar you save will have.

Your Turn: See Study Problem 5.3.

Before you move on to 5.3

Concept Check | 5.2

1. What is compound interest, and how is it calculated?
2. Describe the three basic approaches that can be used to move money through time.
3. How does increasing the number of compounding periods affect the future value of a cash sum?

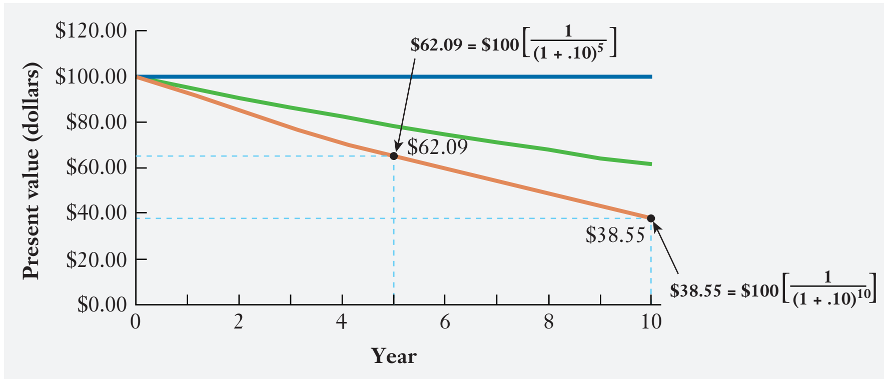
5.3 Discounting and Present Value

So far we have been moving money forward in time; that is, we have taken a known present value of money and determined how much it will be worth at some point in the future. Financial decisions often require calculating the future value of an investment made today. However, there are many instances where we want to look at the reverse question: What is the value today of a sum of money to be received in the future? To answer this question, we now turn our attention to the analysis of present value—the value today of a future cash flow—and the process of **discounting**, determining the present value of an expected future cash flow.

Figure 5.2

The Present Value of $100 Compounded at Different Rates and for Different Time Periods

The present value of $100 to be received in the future becomes smaller as both the interest rate and the number of years rise. At $i = 10\%$, notice that when the number of years goes up from 5 to 10, the present value drops from $62.09 to $38.55.



>> END FIGURE 5.2

The Mechanics of Discounting Future Cash Flows

Discounting is actually the reverse of compounding. We can demonstrate the similarity between compounding and discounting by referring back to the future value formula found in Equation (5–1a):

$$\begin{array}{c}\text{Future Value}\\ \text{in Year } n\\ (FV_n)\end{array} = \begin{array}{c}\text{Present}\\ \text{Value } (PV)\end{array}\left(1 + \begin{array}{c}\text{Annual}\\ \text{Interest Rate } (i)\end{array}\right)^{\text{Number of Years } (n)} \tag{5–1a}$$

To determine the present value of a known future cash flow, we simply take Equation (5–1a) and solve for PV:

$$\begin{array}{c}\text{Present}\\ \text{Value } (PV)\end{array} = \begin{array}{c}\text{Future Value}\\ \text{in Year } n\\ (FV_n)\end{array}\left[\frac{1}{\left(1 + \begin{array}{c}\text{Annual}\\ \text{Interest Rate } (i)\end{array}\right)^{\text{Numbers of Years } (n)}}\right] \tag{5–2}$$

We refer to the term in the brackets as the **present value interest factor**, which is the value by which we multiply the future value to calculate the present value. Thus, to find the present value of a future cash flow, we multiply the future cash flow by the present value interest factor:[2]

$$\begin{array}{c}\text{Present}\\ \text{Value } (PV)\end{array} = \begin{array}{c}\text{Future Value}\\ \text{in year } n\\ (FV_n)\end{array} \times \left(\begin{array}{c}\text{Present Value}\\ \text{Interest Factor}\\ (PVIF)\end{array}\right)$$

where Present Value Interest Factor ($PVIF$) $= \dfrac{1}{(1 + i)^n}$.

Note that the present value of a future sum of money decreases as we increase the number of periods, n, until the payment is received or as we increase the interest rate, i. That, of course, only makes sense because the present value interest factor is the *inverse* of the future value interest factor. Graphically, this relationship can be seen in Figure 5.2. Thus, given a **discount rate**, or interest rate at which money is being brought back to present, of 10 percent, $100 received in 10 years will be worth only $38.55 today. By contrast, if the discount rate is 5 percent, the present value will be $61.39. If the discount rate is

[2] Related tables appear in Appendix C in MyLab Finance.

Checkpoint 5.4

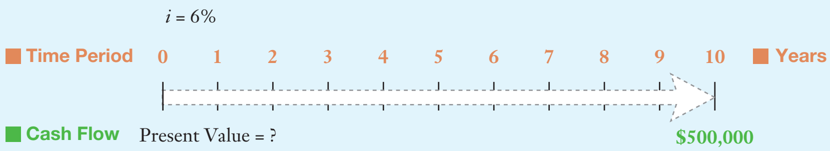
MyLab Finance Video

Solving for the Present Value of a Future Cash Flow

Your firm has just sold a piece of property for $500,000, but under the sales agreement, it won't receive the $500,000 until 10 years from today. What is the present value of $500,000 to be received 10 years from today if the discount rate is 6 percent annually?

STEP 1: Picture the problem

Expressed as a timeline, this problem would look like the following:



STEP 2: Decide on a solution strategy

In this instance, we are simply solving for the present value of $500,000 to be received at the end of 10 years. We can calculate the present value of the $500,000 using Equation (5–2).

STEP 3: Solve

Using the Mathematical Formulas. Substituting $FV_{10} = \$500{,}000$, $n = 10$, and $i = 6\%$ into Equation (5–2), we find

$$\begin{aligned} PV &= \$500{,}000\left[\frac{1}{(1+.06)^{10}}\right] \\ &= \$500{,}000\left[\frac{1}{1.79085}\right] \\ &= \$500{,}000\,[.558394] \\ &= \$279{,}197 \end{aligned}$$

The present value of the $500,000 to be received in 10 years is $279,197. Earlier we noted that discounting is the reverse of compounding. We can easily test this calculation by considering this problem in reverse: What is the future value in 10 years of $279,197 today if the rate of interest is 6 percent? Using our *FV* equation, Equation (5–1a), we can see that the answer is $500,000.

Using a Financial Calculator.

Enter	10	6.0		0	500,000
	N	I/Y	PV	PMT	FV
Solve for			-279,197		

Using an Excel Spreadsheet.

= PV(rate,nper,pmt,fv) or, with values entered, = PV(0.06,10,0,500000)

STEP 4: Analyze

Once you've found the present value of any future cash flow, that present value is in today's dollars and can be compared to other present values. The underlying point of this exercise is to make cash flows that occur in different time periods comparable so that we can make good decisions. Also notice that regardless of which method we use to calculate the future value—computing the formula by hand, with a calculator, or with Excel—we always arrive at the same answer.

STEP 5: Check yourself

What is the present value of $100,000 to be received at the end of 25 years, given a 5 percent discount rate?

ANSWER: $29,530.

Your Turn: For more practice, do related **Study Problems** 5–12, 5–15, 5–19, and 5–28 at the end of this chapter. >> **END Checkpoint 5.4**

10 percent but the \$100 is received in 5 years instead of 10 years, the present value will be \$62.09. This concept of present value plays a central role in the valuation of stocks, bonds, and new proposals. You can easily verify this calculation using any of the discounting methods we describe next.

Two Additional Types of Discounting Problems

Time-value-of-money problems do not always involve calculating either the present value or the future value of a series of cash flows. There are a number of problems that require you to solve for either the number of periods in the future, n, or the rate of interest, i. For example, to answer the following questions, you will need to calculate the number of periods in the future, n:

- How many years will it be before the money I have saved will be enough to buy a second home?
- How long will it take to accumulate enough money for a down payment on a new retail outlet?

And to answer the following questions, you must solve for the interest rate, i:

- What rate do I need to earn on my investment to have enough money for my newborn child's college education (n = 18 years)?
- If our firm introduces a new product line, what interest rate will this investment earn?

Fortunately, with the help of the mathematical formulas, a financial calculator, or an Excel spreadsheet, you can easily solve for i or n in any of these or similar situations.

Solving for the Number of Periods

Suppose you want to know how many years it will take for an investment of \$9,330 to grow to \$20,000 if it's invested at 10 percent annually. Let's take a look at how to solve this using the mathematical formulas, a financial calculator, and an Excel spreadsheet.

Using the Mathematical Formulas. Substituting for FV, PV, and i in Equation (5–1a),

$$\begin{array}{c}\text{Future Value}\\ \text{in Year } n\\ (FV_n)\end{array} = \begin{array}{c}\text{Present}\\ \text{Value } (PV)\end{array}\left(1 + \begin{array}{c}\text{Annual}\\ \text{Interest Rate } (i)\end{array}\right)^{\text{Number of Years } (n)} \quad \textbf{(5–1a)}$$

$$\$20{,}000 = \$9{,}330(1.10)^n$$

Solving for n mathematically is tough. One way is to solve for n using a trial-and-error approach. That is, you could substitute different values of n into the equation—either increasing the value of n to make the right-hand side of the equation larger or decreasing the value of n to make it smaller until the two sides of the equation are equal—but that will be a bit tedious. Using the time-value-of-money features on a financial calculator or in Excel is much easier and faster.

Using a Financial Calculator. Using a financial calculator or an Excel spreadsheet, this problem becomes much easier. With a financial calculator, all you do is substitute in the values for i, PV, and FV and solve for n:

	N	I/Y	PV	PMT	FV
Enter		10.0	-9,330	0	20,000
Solve for	8.0				

You'll notice that PV is input with a negative sign. In effect, the financial calculator is programmed to assume that the \$9,330 is a cash outflow (the money leaving your hands), whereas the \$20,000 is money that you will receive. If you don't give one of these values a negative sign, you can't solve the problem.

Using an Excel Spreadsheet. With Excel, solving for n is straightforward. You simply use = NPER(rate,pmt,pv,fv) or, with variables entered, = NPER(0.10,0,–9330,20000).

Checkpoint 5.5

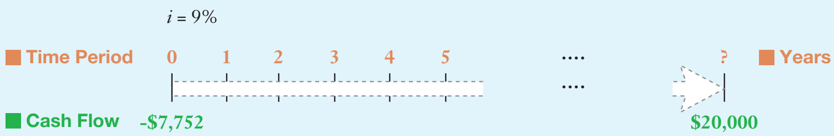
MyLab Finance Video

Solving for the Number of Periods, *n*

Let's assume that the Toyota Corporation has guaranteed that the price of a new Prius will always be $20,000 and that you'd like to buy one but currently have only $7,752. How many years will it take for your initial investment of $7,752 to grow to $20,000 if it is invested so that it earns 9 percent compounded annually?

STEP 1: Picture the problem

In this case, we are solving for the number of periods:



STEP 2: Decide on a solution strategy

In this problem, we know the interest rate, the present value, and the future value, and we want to know how many years it will take for $7,752 to grow to $20,000 at 9 percent interest per year. We are solving for n, and we can calculate it using Equation (5–1a).

STEP 3: Solve

Using a Financial Calculator.

Enter		9.0	-7,752	0	20,000
	N	I/Y	PV	PMT	FV
Solve for	11.0				

Using an Excel Spreadsheet.

= NPER(rate,pmt,pv,fv) or, with values entered, = NPER(0.09,0,–7752,20000)

STEP 4: Analyze

It will take about 11 years for $7,752 to grow to $20,000 at 9 percent compound interest. This is the kind of calculation that both individuals and business make in trying to plan for major expenditures.

STEP 5: Check yourself

How many years will it take for $10,000 to grow to $200,000, given a 15 percent compound growth rate?

ANSWER: 21.4 years.

Your Turn: For more practice, do related **Study Problems** 5–13 and 5–18 at the end of this chapter.

>> END Checkpoint 5.5

The Rule of 72

Now you know how to determine the future value of any investment. What if all you want to know is how long it will take to double your money in that investment? One simple way to approximate how long it will take for a given sum to double in value is called the **Rule of 72**. This "rule" states that you can determine how many years it will take for a given sum to double by dividing the investment's annual growth or interest rate into 72. For example, if an investment grows at an annual rate of 9 percent per year, according to the Rule of 72 it should take $72/9 = 8$ years for that sum to double.

Keep in mind that this is not a hard-and-fast rule, just an approximation—but it's a pretty good approximation. For example, the *future value interest factor* of $(1 + i)^n$ for 8 years ($n = 8$) at 9 percent ($i = 9\%$) is 1.993, which is pretty close to the Rule of 72's approximation of 2.0.

Solving for the Rate of Interest

You have just inherited $34,946 and want to use it to fund your retirement in 30 years. If you have estimated that you will need $800,000 to fund your retirement, what rate of interest will you have to earn on your $34,946 investment? Let's take a look at solving this using the mathematical formulas, a financial calculator, and an Excel spreadsheet to calculate i.

Using the Mathematical Formulas. If you write this problem using our time-value-of-money formula, you get

$$\begin{matrix}\text{Future Value}\\ \text{in Year } n\\ (FV_n)\end{matrix} = \begin{matrix}\text{Present}\\ \text{Value } (PV)\end{matrix}\left(1 + \begin{matrix}\text{Annual}\\ \text{Interest Rate } (i)\end{matrix}\right)^{\text{Number of Years } (n)} \tag{5–1a}$$

$$\$800{,}000 = \$34{,}946(1 + i)^{30}$$

Once again, you could resort to a trial-and-error approach by substituting different values of i into the equation and calculating the value on the right-hand side of the equation to see if it is equal to $800,000. However, again, this would be quite cumbersome and unnecessary. Alternatively, you could solve for i directly by dividing both sides of the equation above by $34,946

$$(1 + i)^{30} = \$800{,}000/\$34{,}946 = 22.8925$$

and then taking the 30th root of this equation to find the value of $(1 + i)$. Because taking the 30th root of something is the same as taking something to the 1/30 (or 0.033333) power, this is a relatively easy process if you have a financial calculator with a "y^n" key. In this case, you (1) enter 22.8925, (2) press the "y^n" key, (3) enter 0.033333, and (4) press the "=" key. The answer should be 1.109999, indicating that $(1 + i) = 1.109999$ and $i = 10.9999\%$ or 11%. As you might expect, it's faster and easier to use the time-value-of-money functions on a financial calculator or in Excel.

Using a Financial Calculator. Using a financial calculator or an Excel spreadsheet, this problem becomes much easier. With a financial calculator, all you do is substitute in the values for n, PV, and FV and solve for i:

	N	I/Y	PV	PMT	FV
Enter	30		-34,946	0	800,000
Solve for		11.0			

Using an Excel Spreadsheet. With Excel, you use = RATE(nper,pmt,pv,fv) or, with values entered, = RATE(30,0,–34946,8000000).

Before you move on to 5.4

Concept Check | 5.3

1. What does the term *discounting* mean with respect to the time value of money?
2. How is discounting related to compounding?

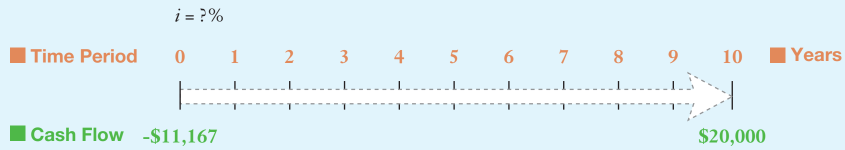
Checkpoint 5.6

Solving for the Interest Rate, *i*

Let's go back to that Prius example in Checkpoint 5.5. Recall that the Prius always costs $20,000. In 10 years, you'd really like to have $20,000 to buy a new Prius, but you have only $11,167 now. At what rate must your $11,167 be compounded annually for it to grow to $20,000 in 10 years?

STEP 1: Picture the problem

We can visualize the problem using a timeline as follows:



STEP 2: Decide on a solution strategy

Here we know the number of years, the present value, and the future value, and we are solving for the interest rate. We'll use Equation (5–1a) to solve this problem.

STEP 3: Solve

Using the Mathematical Formulas.

$$\$20{,}000 = \$11{,}167\,(1 + i)^{10}$$
$$1.7910 = (1 + i)^{10}$$

We then take the 10th root of this equation to find the value of $(1 + i)$. Because taking the 10th root of something is the same as taking something to the 1/10 (or 0.10) power, this can be done if you have a financial calculator with a "y^n" key. In this case, you (1) enter 1.7910, (2) press the "y^n" key, (3) enter 0.10, and (4) press the "=" key. The answer should be 1.06, indicating that $(1 + i) = 1.06$ and $i = 6\%$.

Using a Financial Calculator.

Enter	10		-11,167	0	20,000
	N	I/Y	PV	PMT	FV
Solve for		6.0			

= RATE(nper, pmt, pv, fv) or, with values entered, = RATE(10,0, − 11167,20000)

STEP 4: Analyze

You can increase your future value by growing your money at a higher interest rate or by letting your money grow for a longer period of time. For most of you, when it comes to planning for your retirement, a large n is a real positive for you. Also, if you can earn a slightly higher return on your retirement savings, or any savings for that matter, it can make a big difference.

STEP 5: Check yourself

At what rate will $50,000 have to grow to reach $1,000,000 in 30 years?

ANSWER: 10.5 percent.

Your Turn: For more practice, do related **Study Problems** 5–14, 5–16, 5–17, 5–20 to 5–22, 5–26, and 5–27 at the end of this chapter.

>> END Checkpoint 5.6

5.4 Making Interest Rates Comparable

Sometimes it's difficult to determine exactly how much you are paying or earning on a loan. That's because the loan might be quoted not as compounding annually but rather as compounding quarterly or daily. To illustrate, let's look at two loans, one that is quoted as 8.084 percent compounded annually and one that is quoted as 7.85 percent compounded quarterly. Unfortunately, they are difficult to compare because the interest on one is compounded annually (you pay interest just once a year) but the interest on the other is compounded quarterly (you pay interest four times a year). To allow borrowers to compare rates between different lenders, the U.S. Truth-in-Lending Act requires what is known as the annual percentage rate (APR) to be displayed on all consumer loan documents. The **annual percentage rate (APR)** indicates the interest rate paid or earned in one year without compounding. We can calculate

APR as the interest rate per period (for example, per month or week) multiplied by the number of periods during which compounding occurs during the year (m):

$$\begin{array}{c} \text{Annual Percentage} \\ \text{Rate } (APR) \\ \text{or Simple Interest} \end{array} = \left(\begin{array}{c} \text{Interest Rate per} \\ \text{Period (for example,} \\ \text{per month or week)} \end{array} \right) \times \begin{array}{c} \text{Compounding} \\ \text{Periods per} \\ \text{Year } (m) \end{array} \tag{5–3}$$

Thus, if you are paying 2 percent per month, the number of compounding periods per year (m) would be 12, and the APR would be:

$$APR = 2\%/\text{month} \times 12 \text{ months}/\text{year} = 24\%$$

Unfortunately, the APR does not help much when the rates being compared are not compounded for the same number of periods per year. In fact, the APR is also called the **nominal or quoted (stated) interest rate** because it is the rate that the lender states you are paying.[3] In our example, both 8.084 percent and 7.85 percent are the APRs, but they aren't comparable because the loans have different compounding periods.

To make them comparable, we calculate their equivalent rates using an annual compounding period. We do this by calculating the **effective annual rate (EAR)**, the annual compounded rate that produces the same return as the nominal, or stated, rate. The EAR can be calculated using the following equation:

$$\text{Effective Annual Rate } (EAR) = \left(1 + \frac{\begin{array}{c}\text{APR or Quoted} \\ \text{Annual Rate}\end{array}}{\begin{array}{c}\text{Compounding Periods} \\ \text{per Year } (m)\end{array}} \right)^m - 1 \tag{5–4}$$

We calculate the EAR for the loan that has a 7.85 percent quoted annual rate of interest compounded quarterly (i.e., $m = 4$ times per year) using Equation (5–4) as follows:

$$EAR = \left[1 + \frac{0.0785}{4}\right]^4 - 1 = .08084, \text{ or } 8.084\%$$

So if your banker offers you a loan with a 7.85 percent rate with quarterly compounding or an 8.084 percent rate with annual compounding, which should you prefer? If you didn't know how the time value of money is affected by compounding, you would have chosen the 7.85 percent rate because, on the surface, it looked like the loan with the lower cost. However, you should be indifferent because these two offers have the same cost to you; that is, they have the same EAR. The key point here is that to compare the two loan terms, you need to convert them to the same number of compounding periods (annual, in this case). Given the wide variety of compounding periods used by businesses and banks, it is important to know how to make these rates comparable so you can make logical decisions.

Now let's return to that payday loan we introduced at the beginning of the chapter. What is its EAR? In that example, we looked at a payday lender that advertised that you could borrow \$100 and repay \$126.40 14 days later. On the surface, that looks like you are paying 26.40 percent (\$126.40/\$100 = 1.2640), but that's really what you are paying every fourteen days. To find the quoted annual rate, we multiply the 14-day rate of 26.40 percent times the number of 14-day periods in a year. In this case assuming there are 365 days in a year, m, or the number of 14-day periods in a year is 26.0714 (365 divided by 14). Thus, the quoted annual rate is 0.2640 × 26.0714 = 6.8828 or 688.28%. Substituting into Equation (5–4), we get

$$\begin{aligned} EAR &= \left[1 + \frac{6.8828}{26.0714}\right]^{26.0714} - 1 \\ &= 449.44 - 1 = 448.44, \text{ or } 44{,}844\% \end{aligned}$$

That's just under 450 times what you borrowed—now all that assumes that you get your \$100 and keep on rolling over the loan every two weeks, borrowing the original principal along with all the interest accrued over the previous 14 days. The bottom line is that this is a mighty expensive way to borrow money—and one you should avoid at all costs. And an understanding of the time value of money should help keep you away from payday loans.

[3] Technically, the APR is generally calculated including both interest and fees, while the quoted interest rate only includes interest payments, but for our purposes, they are the same.

Checkpoint 5.7

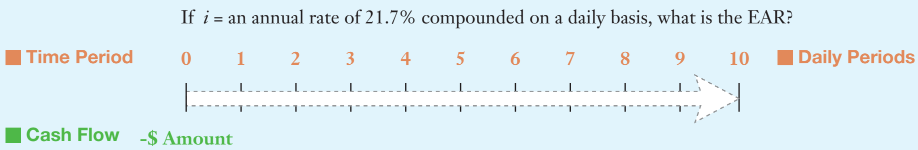
MyLab Finance Video

Calculating an Effective Annual Rate or EAR

Assume that you just received your first credit card statement and the APR, or annual percentage rate, listed on the statement is 21.7 percent. When you look closer, you notice that the interest is compounded daily. What is the EAR, or effective annual rate, on your credit card?

STEP 1: Picture the problem

We can visualize the problem using a timeline as follows:



STEP 2: Decide on a solution strategy

We'll use Equation (5–4) to solve this problem:

$$\begin{array}{c}\text{Effective}\\ \text{Annual Rate}\\ (EAR)\end{array} = \left(1 + \frac{\text{APR or Quoted Annual Rate}}{\text{Compounding Periods per Year } (m)}\right)^m - 1 \qquad (5\text{–}4)$$

STEP 3: Solve

To calculate the EAR, we can use Equation (5–4). Substituting in the quoted annual rate of 21.7 percent, or 0.217 and the *m* of 365, we get

$$EAR = \left[1 + \frac{0.217}{365}\right]^{365} - 1$$
$$= 1.242264 - 1 = 0.242264, \text{ or } 24.2264\%$$

You were right in thinking that the amount of interest you owed seemed high. In reality, the EAR, or effective annual rate, is actually 24.2264 percent. Recall that whenever interest is compounded more frequently, it accumulates faster.

STEP 4: Analyze

When you invest in a certificate of deposit, or CD, at a bank, the rate the bank will quote you is the EAR; that's because it's the rate that you will actually earn on your money—and it's also higher than the simple APR. It's important to make sure when you compare different interest rates that they are truly comparable, and the EAR allows you to make them comparable. For example, if you're talking about borrowing money at 9 percent compounded daily, although the APR is 9 percent, the EAR is actually 9.426 percent. That's a pretty big difference when you're paying the interest.

STEP 5: Check yourself

What is the EAR on a quoted or stated rate of 13 percent that is compounded monthly?

ANSWER: 13.80 percent.

Your Turn: For more practice, do related **Study Problems** 5–35 through 5–38 at the end of this chapter.

>> **END** **Checkpoint 5.7**

Calculating the Interest Rate and Converting It to an EAR

When you have non-annual compounding and you calculate a value for *i* using your financial calculator or Excel, you're calculating the rate per non-annual compounding period, which is referred to as the periodic rate:

$$\text{Periodic Rate} = \frac{\text{APR or Quoted Annual Rate}}{\text{Compounding Periods per Year } (m)}$$

You can easily convert the periodic rate into the APR by multiplying it by the number of times that compounding occurs per year (m). However, if you're interested in the EAR, you'll have to subsequently convert the value you just calculated to an EAR. Let's look at an example.

Suppose that you've just taken out a two-year, $100,000 loan with monthly compounding and that at the end of two years you will pay $126,973 to pay the loan off. How can we find the quoted interest rate on this loan and convert it to an EAR? This problem can be solved using either a financial calculator or Excel.[4] Because the problem involves monthly compounding, the number of compounding periods per year, m, is 12; the number of periods, n, becomes 24 (number of years times m, or 2 times 12); and the solution, i, will be expressed as the *monthly rate*.

Using a Financial Calculator. Substituting in a financial calculator, we find

Enter	24		-100,000	0	126,973
	N	I/Y	PV	PMT	FV
Solve for		1.0			

To determine the APR you're paying on this loan, you need to multiply the value you just calculated for i times 12—thus, the APR on this loan is 12 percent. But that is *not* the loan's EAR; it's merely the APR. To convert the APR to the EAR, we can use Equation (5–4):

$$\text{Effective Annual Rate } (EAR) = \left(1 + \frac{\text{APR or Quoted Annual Rate}}{\text{Compounding Periods per Year } (m)}\right)^{m} - 1 \qquad \textbf{(5–4)}$$

Substituting the quoted annual rate of 0.12 and the m of 12 into the above equation, we get

$$\begin{aligned} EAR &= \left[1 + \frac{0.12}{12}\right]^{12} - 1 \\ &= 1.1268 - 1 = 0.126825, \text{ or } 12.6825\% \end{aligned}$$

In reality, the EAR, or effective annual rate, is actually 12.6825 percent. In effect, if you took out a two-year loan for $100,000 at 12.6825 percent compounded annually, your payment at the end of two years would be $126,973, the same payment you had when you borrowed $100,000 at 12 percent compounded monthly.

To the Extreme: Continuous Compounding

As m (the number of compounding periods per year) increases, so does the EAR. That only makes sense because the greater the number of compounding periods is, the more often interest is earned on interest. As you just saw, we can easily compute the EAR when interest, i, is compounded daily ($m = 365$). We can just as easily calculate the EAR if the interest is compounded every hour ($m = 8{,}760$), every minute ($m = 525{,}600$), or every second ($m = 31{,}536{,}000$). We can even calculate the EAR when interest is continuously compounded—that is, when the time intervals between interest payments are infinitely small:

$$EAR = \left(e^{\text{APR or Quoted Annual Rate}}\right) - 1 \qquad \textbf{(5–5)}$$

[4] The TI BAII-Plus and HP 10BII calculators have a shortcut key that allows you to enter the number of compounding periods and the nominal rate to calculate the EAR. Those keystrokes are shown in Appendix A in MyLab Finance.

Finance in a **Flat World**

Financial Access at Birth

Approximately half the world's population has no access to financial services such as savings, credit, and insurance. UCLA finance professor Bhagwan Chowdhry has a plan called the Financial Access at Birth (FAB) Campaign aimed at eliminating this problem.

This is how FAB could work. Each child would have an online bank account opened at birth with an initial deposit of $100. The bank account would be opened together with the child's birth registration, and the deposit plus interest could be withdrawn when the child reached 16 years of age. If the program were launched in 2016, in just 20 short years every child and young adult in the world would have access to financial services. Assuming a 5 percent annual rate of interest on the $100 deposit, the account would have grown to about $218 when the child reached 16. If we waited until the child reached 21 before turning over the account, it would have grown to about $279. In many parts of the world, this would be a princely sum of money. Moreover, the recipient would have a bank account!

So what's the cost of implementing FAB? Currently, there are about 134 million children born annually, and assuming that a quarter of these children would not need the service, this would leave 100 million children that otherwise would not have access to a bank account. The cost of the program would then be just $10 billion per year, which is less than the amount spent per week on military expenditures around the world. If 100 million individuals would contribute just $100 per year, the dream of the FAB could become a reality. Every person in the world would have access to financial services in just 20 years!

Want to learn more? Go to http://financialaccessatbirth.org.

where e is the number 2.71828, with the corresponding calculator key generally appearing as "e^x." This number e is an irrational number that is used in applications that involve things that grow continuously over time. It is similar to the number Π in geometry.[5]

Let's take another look at the credit card example from Checkpoint 5.7—but with continuous compounding. Again, the APR, or annual percentage rate, is listed at 21.7 percent. With continuous compounding, what's the EAR, or effective annual rate, on your credit card?

$$EAR = e^{.217} - 1 = 1.2423 - 1 = 0.2423, \text{ or } 24.23\%$$

Before you begin end-of-chapter material

Concept Check | 5.4

1. How does an EAR differ from an APR?
2. What is the effect on future values of having multiple compounding periods within a year?

[5] Like the number Π, it goes on forever. In fact, if you're interested, you can find the first 5 million digits of e at http://antwrp.gsfc.nasa.gov/htmltest/gifcity/e.5mil.

Applying the Principles of Finance to Chapter 5

P Principle 1: **Money Has a Time Value** This chapter begins our study of the time value of money—a dollar received today, other things being the same, is worth more than a dollar received a year from now. The concept of the time value of money underlies many financial decisions faced in business. We can calculate the value today of a sum of money received in the future and the future value of a present sum.

Chapter Summaries

5.1 Construct cash flow timelines to organize your analysis of problems involving the time value of money. (pgs. 162–163)

SUMMARY: Timelines can help you visualize and then solve time-value-of-money problems. Time periods—with 0 representing today, 1 the end of Period 1, and so forth—are listed above the timeline. Note that Period 1 represents the end of Period 1 and the beginning of Period 2. The periods can consist of years, months, days, or any unit of time. However, in general, when people analyze cash flows, they are looking at yearly periods. The cash flows appear below the timeline. Cash inflows are labeled with positive signs. Cash outflows are labeled with negative signs.

KEY TERM

Timeline, page 162 A linear representation of the timing of cash flows.

Concept Check | 5.1

1. What is a timeline, and how does it help you solve problems involving the time value of money?
2. Does Year 5 represent the end of the fifth year, the beginning of the sixth year, or both?

5.2 Understand compounding and calculate the future value of cash flows using mathematical formulas, a financial calculator, and an Excel spreadsheet. (pgs. 164–171)

SUMMARY: Compounding begins when the interest earned on an investment during a past period begins earning interest in the current period. Financial managers must compare the costs and benefits of alternatives that do not occur during the same time period. Calculating the time value of money makes all dollar values comparable; because money has a time value, these calculations move all dollar flows either back to the present or out to a common future date. All time value formulas presented in this chapter actually stem from the compounding formula $FV_n = PV(1 + i)^n$. The formulas are used to deal simply with common financial situations—for example, discounting single flows or moving single flows out into the future.

Financial calculators are a handy and inexpensive alternative to doing the math. However, most professionals today use spreadsheet software, such as Excel.

KEY TERMS

Compounding, page 164 The process of determining the future value of a payment or series of payments when applying the concept of compound interest.

Compound interest, page 164 The situation in which interest paid on the investment during the first period is added to the principal and, during the second period, interest is earned on the original principal plus the interest earned during the first period.

Future value, page 164 What a cash flow will be worth in the future.

Future value interest factor, page 165 The value $(1 + i)^n$ used as a multiplier to calculate an amount's future value.

Present value, page 164 The value in today's dollars of a future payment discounted back to the present at the required rate of return.

Simple interest, page 164 The interest earned on the principal.

Concept Check | 5.2

1. What is compound interest, and how is it calculated?
2. Describe the three basic approaches that can be used to move money through time.
3. How does increasing the number of compounding periods affect the future value of a cash sum?

KEY EQUATIONS

$$\begin{matrix}\text{Future Value}\\ \text{in Year } n\\ (FV_n)\end{matrix} = \begin{matrix}\text{Present}\\ \text{Value } (PV)\end{matrix}\left(1+\begin{matrix}\text{Annual}\\ \text{Interest Rate } (i)\end{matrix}\right)^{\text{Number of Years } (n)} \tag{5–1a}$$

$$\begin{matrix}\text{Future Value}\\ \text{in Year } n\\ (FV_n)\end{matrix} = \begin{matrix}\text{Present}\\ \text{Value } (PV)\end{matrix}\left(1+\frac{\begin{matrix}\text{Annual}\\ \text{Interest Rate } (i)\end{matrix}}{\begin{matrix}\text{Compounding}\\ \text{Periods per Year } (m)\end{matrix}}\right)^{m \times (\text{Number of Years } (n))} \tag{5–1b}$$

5.3 Understand discounting and calculate the present value of cash flows using mathematical formulas, a financial calculator, and an Excel spreadsheet. (pgs. 171–176)

SUMMARY: Previously, we were solving for the future value (FV_n) of the present value (PV) of a sum of money. When we are solving for the present value, we are simply doing the reverse of solving for the future value. We can find the present value by solving for PV:

$$PV = FV_n\left[\frac{1}{(1+i)^n}\right]$$

In addition, increasing the number of compounding periods within the year, while holding the rate of interest constant, will magnify the effects of compounding. That is, even though the rate of interest does not change, increasing the number of compounding periods means that interest gets compounded sooner than it would otherwise. This magnifies the effects of compounding.

KEY TERMS

Discounting, page 171 The inverse of compounding. This process is used to determine the present value of a future cash flow.

Discount rate, page 172 The interest rate used in the discounting process.

Present value interest factor, page 172 The value $[1/(1+i)^n]$ used as a multiplier to calculate a future payment's present value.

Rule of 72, page 175 A method for estimating the time it takes for an amount to double in value. To determine the approximate time it takes for an amount to double in value, 72 is divided by the annual interest rate.

KEY EQUATIONS

$$\begin{matrix}\text{Present}\\ \text{Value } (PV)\end{matrix} = \frac{\begin{matrix}\text{Future Value}\\ \text{in Year } n \ (FV_n)\end{matrix}}{\left(1+\begin{matrix}\text{Annual}\\ \text{Interest Rate } (i)\end{matrix}\right)^{\text{Number of year } (n)}}$$

$$PV = \begin{matrix}\text{Future Value}\\ \text{in Year } n\\ (FV_n)\end{matrix}\left[\frac{1}{\left(1+\frac{\begin{matrix}\text{Annual}\\ \text{Interest Rate } (i)\end{matrix}}{\begin{matrix}\text{Compounding}\\ \text{Periods per Year } (m)\end{matrix}}\right)^{m \times (\text{Numbers of Years } (n))}}\right] \tag{5–2}$$

Concept Check | 5.3

1. What does the term *discounting* mean with respect to the time value of money?
2. How is discounting related to compounding?

5.4 Understand how interest rates are quoted and know how to make them comparable. (pgs. 177–181)

SUMMARY: One way to compare different interest rates is to use the annual percentage rate (APR), which indicates the amount of interest earned in one year without compounding. The APR is the simple interest rate and is calculated as the interest rate per period multiplied by the number of periods in the year:

$$APR = \text{Interest Rate per Period} \times \text{Periods per Year} \tag{5–3}$$

The problem with the APR occurs if compounding occurs more than once a year—for example, if the interest you owe is calculated every month, then in the second month, and from then on, you will end up paying interest from the first month. The end result of this is that the actual interest rate you are paying is greater than the APR. To find out the actual amount of interest we would pay over the course of one time period, we must convert the quoted APR rate to an effective annual rate (EAR). The EAR is the annual compounded rate that produces the same cash flow as the nominal interest rate:

$$EAR = \left(1 + \frac{\text{APR or Quoted Annual Rate}}{m}\right)^m - 1 \qquad \textbf{(5–4)}$$

where m is the number of compounding periods within a year.

KEY TERMS

Annual percentage rate (APR), page 177 The interest rate paid or earned in one year without compounding. It is calculated as the interest rate per period (for example, per month or week) multiplied by the number of periods during which compounding occurs during the year (m).

Effective annual rate (EAR), page 178 The annual compounded rate that produces the same return as the nominal, or stated, rate.

Nominal or quoted (stated) interest rate, page 178 The same as the APR. Technically, the APR is generally calculated including both interest and fees, while the quoted interest rate only includes interest payments, but for our purposes, they are the same.

KEY EQUATIONS

$$\begin{array}{c}\text{Annual Percentage}\\ \text{Rate }(APR)\\ \text{or Simple Interest}\end{array} = \left(\begin{array}{c}\text{Interest}\\ \text{Rate per}\\ \text{Period}\end{array}\right) \times \begin{array}{c}\text{Compounding}\\ \text{Periods per}\\ \text{Year}\end{array} \qquad \textbf{(5–3)}$$

$$\text{Effective Annual Rate }(EAR) = \left(1 \frac{\begin{array}{c}\text{Quoted}\\ \text{Annual Rate}\end{array}}{\begin{array}{c}\text{Compounding Periods}\\ \text{per Year }(m)\end{array}}\right)^m -1 \qquad \textbf{(5–4)}$$

Concept Check | 5.4

1. How does an EAR differ from an APR?
2. What is the effect on future values of having multiple compounding periods within a year?

Study Questions

5–1. What is the time value of money? Give three examples of how the time value of money might take on importance in business decisions.

5–2. The processes of discounting and compounding are related. Explain this relationship.

5–3. What is the relationship between the number of times interest is compounded per year on an investment and the future value of that investment? What is the relationship between the number of times compounding occurs per year and the EAR?

5–4. How would an increase in the interest rate (i) or a decrease in the number of periods (n) affect the future value (FV_n) of a sum of money?

5–5. How would an increase in the interest rate (i) or a decrease in the number of periods until the payment is received (n) affect the present value (PV) of a sum of money?

5–6. Compare some of the different financial calculators that are available on the Internet. Look at Kiplinger Online calculators (www.kiplinger.com/tools/index.html), which include how much you need to retire, the value of boosting your 401(k) contributions, and how much you can save by biking to work. Also go to www.dinkytown.net, www.bankrate.com/calculators.aspx, and www.interest.com, and click on the “Calculators” links. Which financial calculators do you find to be the most useful? Why?

5–7. In the *Payday Loans* feature on page 161, we examined these short-term, high-interest loans. Recently, Congress passed legislation limiting the interest rate charged to active military to 36 percent. Go to the Predatory Lending Association website at www.predatorylendingassociation.com, find the military base closest to you, and identify the payday lenders that surround that base. Also identify any payday lenders near you.

5–8. In the *Payday Loans* feature on page 161, we examined these short-term, high-interest loans. Go to the Responsible Lending Organization website at www.responsiblelending.org/payday-lending/. What are some of their concerns about payday and other small dollar loans? Summarize their research papers.

Study Problems

MyLab Finance
Go to **www.myfinancelab.com** to complete these exercises online and get instant feedback.

Compound Interest

5–1. **(Calculating future value)** **(Related to Checkpoint 5.2 on page 168)** To what amount will the following investments accumulate?

a. $5,000 invested for 10 years at 10 percent compounded annually
b. $8,000 invested for 7 years at 8 percent compounded annually
c. $775 invested for 12 years at 12 percent compounded annually
d. $21,000 invested for 5 years at 5 percent compounded annually

5–2. **(Calculating future value)** **(Related to Checkpoint 5.2 on page 168)** Leslie Mosallam, who recently sold her Porsche, placed $10,000 in a savings account paying annual compound interest of 6 percent.

a. Calculate the amount of money that will accumulate if Leslie leaves the money in the bank for 1, 5, and 15 years.
b. Suppose Leslie moves her money into an account that pays 8 percent or one that pays 10 percent. Rework part a using 8 percent and 10 percent.
c. What conclusions can you draw about the relationship among interest rates, time, and future sums from the calculations you just did?

5–3. **(Calculating future value)** **(Related to *Finance for Life: Saving for Your First House* on page 171)** You are hoping to buy a house in the future and recently received an inheritance of $20,000. You intend to use your inheritance as a down payment on your house.

a. If you put your inheritance in an account that earns a 7 percent interest rate compounded annually, how many years will it be before your inheritance grows to $30,000?
b. If you let your money grow for 10.25 years at 7 percent, how much will you have?
c. How long will it take your money to grow to $30,000 if you move it into an account that pays 3 percent compounded annually? How long will it take your money to grow to $30,000 if you move it into an account that pays 11 percent?
d. What does all of this tell you about the relationship among interest rates, time, and future sums?

5–4. **(Calculating future value)** **(Related to Checkpoint 5.2 on page 168)** John Meyers recently graduated from university and has managed to find a job. He is from an economically weak community and has received help from a local charity supported by the city council. John has decided to donate one month's salary to the charity. However, as the charity is well funded right now, he has decided to deposit his one month's salary, £1,800, into an account paying 3.67 percent on the condition that the city cannot collect any money from that account for next 120 years. How much money will the city receive from John's donation in 120 years' time?

5–5. **(Calculating compound interest with non-annual periods)** **(Related to Checkpoint 5.3 on page 170)** Calculate the amount of money that will be in each of the following accounts at the end of the given deposit period:

Account Holder	Amount Deposited	Annual Interest Rate	Compounding Periods per Year (*m*)	Compounding Periods (years)
Theodore Logan III	$ 1,000	10%	1	10
Vernell Coles	95,000	12	12	1
Tina Elliott	8,000	12	6	2
Wayne Robinson	120,000	8	4	2
Eunice Chung	30,000	10	2	4
Kelly Cravens	15,000	12	3	3

5–6. **(Calculating compound interest with non-annual periods)** **(Related to Checkpoint 5.2 on page 168)** You just received a $5,000 bonus.

a. Calculate the future value of $5,000, given that it will be held in the bank for five years and earn an annual interest rate of 6 percent.

b. Recalculate part a using a compounding period that is (1) semiannual and (2) bimonthly.

c. Recalculate parts a and b using a 12 percent annual interest rate.

d. Recalculate part a using a time horizon of 12 years at a 6 percent interest rate.

e. What conclusions can you draw when you compare the answers to parts c and d with the answers to parts a and b?

5–7. **(Calculating compound interest with non-annual periods)** **(Related to Checkpoint 5.3 on page 170)** Your grandmother just gave you $6,000. You'd like to see how much it might grow if you invest it.

a. Calculate the future value of $6,000, given that it will be invested for five years at an annual interest rate of 6 percent.

b. Recalculate part a using a compounding period that is (1) semiannual and (2) bimonthly.

c. Now let's look at what might happen if you can invest the money at a 12 percent rate rather than a 6 percent rate; recalculate parts a and b for a 12 percent annual interest rate.

d. Now let's see what might happen if you invest the money for 12 years rather than 5 years; recalculate part a using a time horizon of 12 years (the annual interest rate is still 6 percent).

e. With respect to the changes in the stated interest rate and the length of time the money is invested in parts c and d, what conclusions can you draw?

5–8. **(Calculating future value)** **(Related to Checkpoint 5.2 on page 168)** Parisian Window Ltd. is a new business that has just booked €140,000 in sales in its first year of trading. The managers expect that they will achieve 25 percent sales growth per year. What will their expected sales in Years 2, 3, and 4 be?

5–9. **(Calculating future value)** **(Related to Checkpoint 5.2 on page 168)** You have just introduced "must-have" headphones for the iPod. Sales of the new product are expected to be 10,000 units this year and to increase by 15 percent per year in the future. What are expected sales during each of the next three years? Graph this sales trend, and explain why the number of additional units sold increases every year.

5–10. **(Calculating future value)** **(Related to Checkpoint 5.2 on page 168)** Penny has just turned 35 years old and plans to retire when she turns 65. She wants to buy a small cottage in the Alps after retirement. She has just deposited €25,000 in an account that pays 6 percent. What amount will she get upon retirement? How much will she get if she decides to retire five years earlier, at the age of 60?

5–11. **(Calculating simple and compound interest)** **(Related to Checkpoint 5.2 on page 168)** Jake Mai deposited ¥300,000 in a bank account earning 9 percent per annum. How much interest will he earn in the fourth year? How much of the total will be simple interest and how much will be the result of compound interest?

Discounting and Present Value

5–12. **(Calculating present value)** **(Related to Checkpoint 5.4 on page 173)** Thomas Hill would like to have £1,500,000 at the time of his retirement, which is due in 40 years' time. He has found a fixed income fund that pays 5 percent per annum. How much does he have to invest today? If he invests in a tracker fund that pays 15 percent per annum, how quickly can he retire?

5–13. **(Solving for *n*)** **(Related to Checkpoint 5.5 on page 175)** How many years will the following take?

a. \$500 to grow to \$1,039.50 if it's invested at 5 percent compounded annually

b. \$35 to grow to \$53.87 if it's invested at 9 percent compounded annually

c. \$100 to grow to \$298.60 if it's invested at 20 percent compounded annually

d. \$53 to grow to \$78.76 if it's invested at 2 percent compounded annually

5–14. **(Solving for *i*)** **(Related to Checkpoint 5.6 on page 176)** At what annual interest rate would the following have to be invested?

a. \$500 to grow to \$1,948.00 in 12 years

b. \$300 to grow to \$422.10 in 7 years

c. \$50 to grow to \$280.20 in 20 years

d. \$200 to grow to \$497.60 in 5 years

5–15. **(Calculating present value)** **(Related to Checkpoint 5.4 on page 173)** What is the present value of the following future amounts?

a. \$800 to be received 10 years from now discounted back to the present at 10 percent

b. \$300 to be received 5 years from now discounted back to the present at 5 percent

c. \$1,000 to be received 8 years from now discounted back to the present at 3 percent

d. \$1,000 to be received 8 years from now discounted back to the present at 20 percent

5–16. **(Solving for *i*)** **(Related to Checkpoint 5.6 on page 176)** Kirk Van Houten, who has been married for 23 years, would like to buy his wife an expensive diamond ring with a platinum setting on their 30-year wedding anniversary. Assume that the cost of the ring will be \$12,000 in seven years. Kirk currently has \$4,510 to invest. What annual rate of return must Kirk earn on his investment to accumulate enough money to pay for the ring?

5–17. **(Solving for *i*)** **(Related to Checkpoint 5.6 on page 176)** You are considering investing in a security that will pay you \$1,000 in 30 years.

a. If the appropriate discount rate is 10 percent, what is the present value of this investment?

b. Assume these securities sell for \$365, in return for which you receive \$1,000 in 30 years. What is the rate of return investors earn on this security if they buy it for \$365?

5–18. **(Solving for *n*)** **(Related to Checkpoint 5.5 on page 175)** Alena has always wanted to own a Porsche 911 R sports car. She knows that this car is available in the market for £350,000 and sells higher than its cost value as only a limited number of models were produced. She has invested £42,000 in a fund that pays 5 percent interest per annum. How long does she have to wait before she can get her dream car?

5–19. **(Calculating present value)** **(Related to Checkpoint 5.4 on page 173)** Yassir Ismail has recently bought some land from a farmer in Africa. The farmer has asked that, in addition to immediate payment, Yassir will also pay \$500,000 in 30 years' time per local customs related to ownership transfers to secure earnings for the next generation. If the appropriate discount rate is 6 percent, what is the present value of this future payment?

5–20. **(Solving for *i*)** **(Related to Checkpoint 5.6 on page 176)** Dave Kaminsky bought a Ford Mustang seven years ago for £32,000. He has decided to sell it as he now needs a pick-up truck for his new business. He received £12,000 for it from a dealer in part exchange, which means that since the price of the Ford Mustang was more than that of the pick-up truck, Dave received £12,000 and the pick-up truck in exchange for his Ford Mustang. What was his rate of return for this transaction?

5–21. **(Solving for *i*)** **(Related to Checkpoint 5.6 on page 176)** Springfield Learning sold zero-coupon bonds (bonds that don't pay any interest—instead, the bondholder gets just one payment, coming when the bond matures, from the issuer) and received $900 for each bond that will pay $20,000 when it matures in 30 years.

a. At what rate is Springfield Learning borrowing the money from investors?

b. If Nancy Muntz purchased a bond at the offering for $900 and sold it 10 years later for the market price of $3,500, what annual rate of return did she earn?

c. If Barney Gumble purchased Muntz's bond at the market price of $3,500 and held it 20 years until maturity, what annual rate of return did he earn?

5–22. **(Solving for *i*)** **(Related to Checkpoint 5.6 on page 176)** A business offer promises to pay €22,000 in 12 years' time. You are required to invest €1,600 now to avail this offer. What annual rate of interest would you earn if you took the offer?

5–23. **(Solving for *i*)** A business partner of yours has borrowed £8,000 from you today and has promised to give you £25,000 in 10 years. What annual rate of interest are you going to earn on this loan?

5–24. **(Solving for *n* with non-annual periods)** Assume that government treasury bonds in India pay 14 percent compounded semi-annually. How long will it take for an investment to double itself?

5–25. **(Solving for *n* with non-annual periods)** How long would it take an investment to grow five-fold if it were invested at 8 percent compounded quarterly?

5–26. **(Solving for *i*)** **(Related to Checkpoint 5.6 on page 176)** You lend a friend $10,000, for which your friend will repay you $27,027 at the end of five years. What interest rate are you charging your "friend"?

5–27. **(Solving for *i*)** **(Related to Checkpoint 5.6 on page 176)** You've run out of money for college, and your college roommate has an idea for you. He offers to lend you $15,000, for which you will repay him $37,313 at the end of five years. If you took this loan, what interest rate would you be paying on it?

5–28. **(Comparing present and future values)** **(Related to Checkpoint 5.4 on page 173)** You are offered $100,000 today or $300,000 in 13 years. Assuming that you can earn 11 percent on your money, which should you choose?

5–29. **(Comparing present and future values)** Greer Hill just had a meeting with the pension advisor of her company. She was presented with three options for withdrawing her contribution so far. If she withdraws now, she will get £15,000 immediately. If she withdraws in 10 years' time, she will get £28,000; and if she withdraws after 22 years, she will get £40,000. Assuming she earns 8 percent on her investments, which option should she choose?

5–30. **(Solving for *i*)** **(Related to Checkpoint 5.6 on page 176)** In September 1963, the first issue of the comic book *The X-Men* was issued. The original price for the issue was 12 cents. By September 2016, 53 years later, given the condition it's in, the value of this comic book had risen to $14,500. What annual rate of interest would you have earned if you had bought the comic in 1963 and sold it in 2016?

5–31. **(Solving for *i*)** In March 1963, Iron Man was introduced in issue 39 of the comic book *Tales of Suspense*. The original price for that issue was 12 cents. By March 2016, 53 years later, the value of this comic book, given the condition it's in, had risen to $10,000. What annual rate of interest would you have earned if you had bought the comic in 1963 and sold it in 2016?

5–32. **(Solving for *i*)** A new business proposal is asking you to invest $30,000 now with a guaranteed return of $320,000 in 30 years' time. What annual rate of return would you earn if you invested in this business?

5–33. **(Solving a spreadsheet problem)** If you invest $900 in a bank where it will earn 8 percent compounded annually, how much will it be worth at the end of seven years? Use a spreadsheet to calculate your answer.

5–34. **(Solving a spreadsheet problem)** In 20 years, you would like to have $250,000 to buy a vacation home. If you have only $30,000, at what rate must it be compounded annually for it to grow to $250,000 in 20 years? Use a spreadsheet to calculate your answer.

Making Interest Rates Comparable

5–35. **(Calculating an EAR)** **(Related to Checkpoint 5.7 on page 179)** After examining the various personal loan rates available to you, you find that you can borrow funds from a finance company at 12 percent compounded monthly or from a bank at 13 percent compounded annually. Which alternative is the more attractive?

5–36. **(Calculating an EAR)** **(Related to Checkpoint 5.7 on page 179)** You have a choice of borrowing money from a finance company at 24 percent compounded monthly or from a bank at 26 percent compounded annually. Which alternative is the more attractive?

5–37. **(Calculating an EAR)** **(Related to Checkpoint 5.7 on page 179)** Lisi Jiang needs to renovate her house. She has loan offers from two banks. Bank A is charging 13 percent interest compounded semiannually, while Bank B is charging 12 percent compounded monthly. Which of these loans would you recommend to Lisi?

5–38. **(Calculating an EAR)** **(Related to Checkpoint 5.7 on page 179)** Royal Leeds Bank gives 5 percent interest on a fixed deposit compounded annually. The Citizen York Bank offers 4.75 percent on its fixed deposit compounded monthly. Where would you prefer to deposit your money?

5–39. **(Calculating an EAR)** If you borrow £100 from a payday lender in the United Kingdom, you are expected to pay back £120 in 15 days. What is the effective annual rate (or annual percentage rate) on this type of loan?

5–40. **(Calculating an EAR)** In early 2016, typical terms on a payday loan involved a $15 charge for a two-week payday loan of $100. Assuming there are 26 fourteen-day periods in a year, what is the effective annual rate on such a loan?

Mini-Case

Cameron Williams, 25, is a very organized person. He has recently joined his first full-time job, which pays him £35,000 per annum, and now wants to ensure that he saves for all his future life goals. His employer participates in a workplace pension scheme and matches up to 4 percent of salary contribution toward the pension. Cameron also wants to buy a house as soon as possible and feels that he will need £20,000 as a down payment toward his first home. He has been working part-time from a very young age and has saved a total of £30,000, which has been invested in government bonds paying 3 percent per annum.

He also wants to go on a round-the-world trip before he turns 30. He has estimated that this will cost him around £22,000. He expects to save £8,000 in the next three years and take a personal loan for the remaining amount to achieve his dream holiday plan.

Questions

1. Explain to Cameron how much an investment of £10,000 will grow in 40 years if it earns 8 percent per annum and advise if it's beneficial for him to join his pension plan as early as possible.
2. Assuming that he can earn 6 percent on his savings, how much does Cameron need to deposit now to cover the amount of his contribution in three years' time? How much will he need if he can earn 11 percent on his savings?
3. What will be the value of his savings if he withdraws £20,000 for his house purchase when he is 30 and leaves the remaining in the same investment till he retires at the age of 65?
4. How are compounding and discounting related?
5. Suggest two ways by which Cameron can have a higher amount available to him at the time of his retirement.

Part 1 Introduction to Financial Management (Chapters 1, 2, 3, 4)

Part 2 **Valuation of Financial Assets (Chapters 5, 6, 7, 8, 9, 10)**

Part 3 Capital Budgeting (Chapters 11, 12, 13, 14)

Part 4 Capital Structure and Dividend Policy (Chapters 15, 16)

Part 5 Liquidity Management and Special Topics in Finance (Chapters 17, 18, 19, 20)

CHAPTER 6

The Time Value of Money
Annuities and Other Topics

Chapter **Outline**

6.1 Annuities **(pgs. 192–205)** → **Objective 1.** Distinguish between an ordinary annuity and an annuity due, and calculate the present and future values of each.

6.2 Perpetuities **(pgs. 205–207)** → **Objective 2.** Calculate the present value of a level perpetuity and a growing perpetuity.

6.3 Complex Cash Flow Streams **(pgs. 208–211)** → **Objective 3.** Calculate the present and future values of complex cash flow streams.

Principles P1 and P3 Applied

In this chapter, we continue our examination of the first principle of finance—P Principle 1: **Money Has a Time Value**. Here we provide you with tools that allow you to move cash flow streams through time, determining the value of those streams that have a limited life and those that continue forever and have no maturity date. Once you've mastered these tools, you'll be applying them to the valuation of stocks, bonds, and other investment opportunities in addition to using them to determine your mortgage and car loan payments. The basis for our focus on cash flows is found in P Principle 3: **Cash Flows Are the Source of Value**.

Many of you have bought a car that was financed by a bank. To repay such a loan, you have made payments to the bank of a certain fixed amount each month for 48 or 60 months. Similarly, if you buy a house that you finance with a conventional mortgage, you again face a schedule of fixed payments due over a set period of time. Just like individuals, firms make loan payments that are due at regular intervals. In addition, they pay a fixed amount over a set period of time to lease equipment. And they often have investments that generate regular payments of cash. These examples all have one thing in common—they each require a fixed cash flow stream over a set period of time. As we'll see in this chapter, we call this type of cash flow stream an *annuity.*

Regardless of Your Major…

"Annuities We All Know"

We encounter annuities often in our day-to-day lives. An annuity, as defined in this chapter, is simply a series of equal payments, each payable at the beginning or end of each period (month or year) and over multiple periods. For example, if you're paying off a student loan, you're paying off an annuity. In this case, the annuity represents the payment of principal and interest on your student loan. So if you have $30,000 in student loans outstanding at a 6.8 percent interest rate that you plan to repay in 10 years, you'll be making monthly payments of $345.24 over the next 10 years. We sometimes encounter annuities in which the payments are made to us. For example, if your grandparents leave you $30,000 to help pay your college expenses, you might purchase a 6 percent annuity that provides you with monthly payments of $704.55 over the next four years.

Your Turn: See Study Question 6–1.

6.1 Annuities

In Chapter 5, we learned how to move single cash flows through time, calculating their future and present values. We will now extend these formulas to find the future and present values of a constant stream of cash flows. Together with what we learned in Chapter 5, the material in this chapter provides us with the tools to implement Principle 1: **Money Has a Time Value**. Later in the book, we will see that this principle, along with Principle 3: **Cash Flows Are the Source of Value**, provides the logic behind the valuation of stocks, bonds, and other investment opportunities.

Ordinary Annuities

We define an **annuity** as a series of *equal* dollar payments that are made at the end of equidistant points in time (such as monthly, quarterly, or annually) over a finite period of time (such as three years). Payments for an annuity can be made at either the beginning or the end of each period. If payments are made at the end of each period, the annuity is often referred to as an **ordinary annuity**. An ordinary annuity is the most common form of annuity and is oftentimes referred to simply as an annuity, without the term *ordinary* preceding it. However, some annuities have payments that are made at the beginning of each period, such as apartment rent. We'll discuss this type of annuity later in this chapter. For now, when we refer to an annuity, you should assume we are referring to an ordinary annuity where

- the payments are made at the end of each period;
- the periods are equidistant in time, such as monthly or annually; and
- the payments are made for a finite period of time, such as three years.

The present and future values of an ordinary annuity can be computed using the methods described in Chapter 5 and illustrated in Figure 6.1. However, this process can be time-consuming, especially for longer annuities, so next we will discuss some simple time-value-of-money formulas for easily calculating the present and future values of an annuity.

The Future Value of an Ordinary Annuity

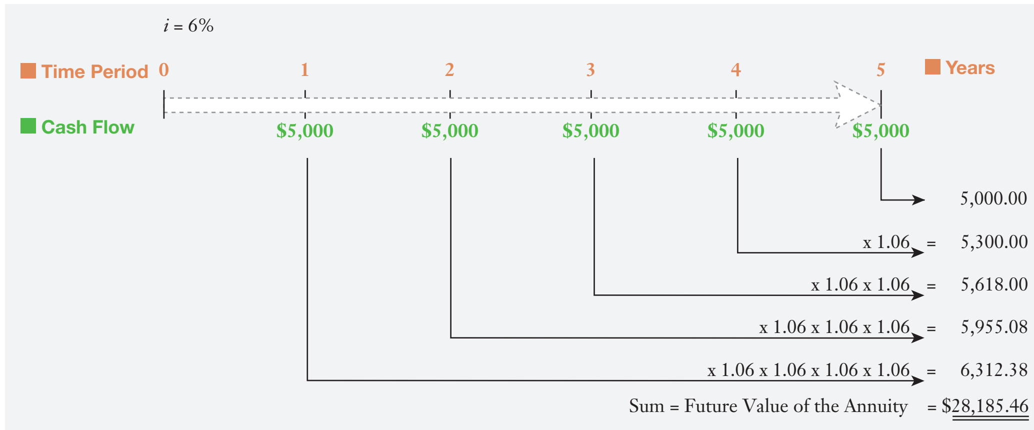
Let's assume that you are saving money to go to graduate school. You've taken your first job, and you plan to save $5,000 each year for the next five years for your grad school fund. How much money will you accumulate by the end of Year 5?

This scenario presents a common annuity valuation problem, one we can solve by finding the future value. Before we can do that, however, we need to know two things: first, the rate of interest you will earn on your savings and, second, the length of time each of your

Figure 6.1

Future Value of a Five-Year Annuity—Saving for Grad School

The five annual annuity payments consist of $5,000 in savings that is placed in an account that earns 6% interest. This is a five-year ordinary annuity, as the first cash flow occurs at the *end* of the first year. This, in turn, means that this payment is compounded only until the end of Year 5 (or for four years).



>> **END FIGURE 6.1**

savings deposits (the annuity payments) will earn interest. For our purposes, let's assume you save $5,000 each year for five years and that you deposit that money in an account that earns 6 percent interest per year. A timeline depicting this is shown in Figure 6.1. We can use Equation (5–1a) from the last chapter to find the future value of each of the deposits. The future value at the end of Year *5* of the deposit made at the end of Year *1* (growing for four years) can be calculated as follows:

$$FV_n = PV(1 + i)^n \tag{5–1a}$$

$$FV_{\text{Year 4}} = \$5{,}000(1 + .06)^4 = \$6{,}312.38$$

Note that the deposit made at the end of Year 1 has only four years to earn interest—until the end of Year 5. Similarly, the deposit made at the end of Year 2 has only three years to earn interest, and so forth. The future value of the first- through fifth-year deposits is as follows:

Interest Rate = 6%
Annuity Payment = $5,000

Year	Payment	Future Value at the End of Year 5
1	$5,000	$ 6,312.38
2	$5,000	$ 5,955.08
3	$5,000	$ 5,618.00
4	$5,000	$ 5,300.00
5	$5,000	$ 5,000.00
Sum = Future Value of Annuity =		$28,185.46

What this calculation tells us is that, if all goes as planned, by the end of five years you should have saved a total of $28,185.47 to help fund graduate school.

Figure 6.1 illustrates the computation of a future value of an annuity using a timeline. It's important to note that the future value of an annuity is simply the sum of the future values of each of the annuity payments compounded out to the end of the annuity's life—in this case, the end of Year 5.

The Formula for the Future Value of an Ordinary Annuity

We can solve for the future value of an ordinary annuity using the following equation:

$$\text{Future Value of an Annuity} = \text{Sum of the Future Values of the Individual Cash Flows That Make Up the Annuity}$$

$$\text{Future Value of an Annuity } (FV_n) = \text{Annuity Payment}\left(1 + \text{Interest Rate}\right)^{n-1} + \text{Annuity Payment}\left(1 + \text{Interest Rate}\right)^{n-2} + \cdots + \text{Annuity Payment}\left(1 + \text{Interest Rate}\right)^{0}$$

Note that there are n payments in the ordinary annuity. However, because the first payment is received at the end of the first period, it is compounded for only $n - 1$ periods until the end of the nth period. In addition, the last payment is received at the end of the nth period so it is compounded for $n - n$, or zero, periods. Using symbols,

$$FVn = PMT(1 + i)^{n-1} + PMT(1 + i)^{n-2} + \ldots + PMT(1 + i)^{1} + PMT(1 + i)^{0} \quad \textbf{(6–1)}$$

Important Definitions and Concepts:

- FV_n is the future value of the annuity at the end of the nth period. Thus, if periods are measured in years, the future value at the end of the third year is FV_3.
- PMT is the annuity payment deposited or received at the end of each period.
- i is the interest (or compound) rate per period. Thus, if the periods are measured in years, it is the annual interest rate.
- n is the number of periods for which the annuity will last.

If we just factor out the PMT term in Equation (6–1), we get the following expression:

$$FV_n = PMT\left[(1 + i)^{n-1} + (1 + i)^{n-2} + \cdots + (1 + i)^{1} + (1 + i)^{0}\right] \quad \textbf{(6–1a)}$$

The sum found in brackets is commonly referred to as the **annuity future value interest factor**.[1] This sum can be reduced to the following expression:

$$\text{Annuity Future Value Interest Factor} = \left[\frac{(1 + i)^{n} - 1}{i}\right] \quad \textbf{(6–1b)}$$

So to calculate the future value of an ordinary annuity of n years where the individual payments are compounded at a rate i, we simply multiply the payment by the annuity future value interest factor:

$$FV_n = PMT\left[\frac{(1 + i)^{n} - 1}{i}\right] \quad \textbf{(6–1c)}$$

Using the Mathematical Formulas. Continuing with our saving-for-grad-school example, we note that PMT is \$5,000 per year, i is 6 percent annually, and n is five years. Thus,

$$FV_n = PMT\left[\frac{(1 + i)^{n} - 1}{i}\right] = \$5{,}000\left[\frac{(1 + .06)^{5} - 1}{.06}\right] = \$5{,}000(5.63709296) = \$28{,}185.46$$

By simply substituting the values given above for PMT, i, and n into Equation (6–1c), we compute the future value of the level payment annuity with one computation rather than five separate future value computations that must then be summed.

Using a Financial Calculator.

Enter	5	6.0	0	-5,000	
	N	I/Y	PV	PMT	FV
Solve for					28,185.46

[1] Annuity future value interest factors can be found in Appendix D in MyLab Finance.

Using an Excel Spreadsheet.

$$= FV(\text{rate,nper,pmt,pv}) \text{ or, with values entered, } = \text{FV}(.06,5,-50{,}000,)$$

Solving for the Payment in an Ordinary Annuity

Instead of figuring out how much money you will accumulate if you deposit a steady amount of money in a savings account each year, perhaps you would like to know how much money you *need to save* each year to accumulate a certain amount by the end of n years. In this case, we know the values of n, i, and FV_n in Equation (6–1c); what we do not know is the value of PMT, the annuity payment deposited each period.

Let's look at an example where you are trying to find out how much you must deposit annually in an account in order to reach a set goal. Suppose that you would like to have \$50,000 saved 10 years from now to finance your goal of getting an MBA. If you are going to make equal annual end-of-year payments to an investment account that pays 8 percent, how big do these annual payments need to be?

Here, using Equation (6–1c), we know that $n = 10$, $i = 8$, and $FV_{10} = \$50{,}000$, but what we do not know is the value of PMT, the annuity payment deposited each year. Substituting into Equation (6–1c), we get

$$\begin{aligned} FV_n &= PMT\left[\frac{(1+i)^n - 1}{i}\right] \\ \$50{,}000 &= PMT\left[\frac{(1+.08)^{10} - 1}{.08}\right] \\ \$50{,}000 &= PMT(14.4866) \\ \frac{\$50{,}000}{14.4866} &= PMT = \$3{,}451 \end{aligned} \tag{6–1c}$$

Checkpoint 6.1 demonstrates the calculation of an annuity payment using the mathematical formulas, a financial calculator, and Excel.

Solving for the Interest Rate in an Ordinary Annuity

You may also want to calculate the *interest rate* you must earn on your investment that will allow your savings to grow to a certain amount of money by a certain future date. In this case, you will be solving for i. Consider the following example: In 15 years, you hope to have \$30,000 saved to buy a sports car. You will be able to save \$1,022 at the end of each year for the next 15 years. What rate of return must you earn on your investments in order to achieve your goal?

It is easy to solve this problem with either a financial calculator or Excel, but as we describe next, solving it with mathematical formulas can be somewhat difficult.

Using the Mathematical Formulas. Substituting the numbers into Equation (6–1c), we get

$$\begin{aligned} FV_n &= PMT\left[\frac{(1+i)^n - 1}{i}\right] \\ \$30{,}000 &= \$1{,}022\left[\frac{(1+i)^{15} - 1}{i}\right] \\ \frac{\$30{,}000}{\$1{,}022} &= \left[\frac{(1+i)^{15} - 1}{i}\right] \\ 29.354 &= \left[\frac{(1+i)^{15} - 1}{i}\right] \end{aligned} \tag{6–1c}$$

The only way to solve for the interest rate at this point is by trial and error. Specifically, we substitute different numbers for i until we find the value of i that makes the right-hand side of the expression equal to 29.354.

Using a Financial Calculator. We can use a financial calculator to solve for i directly as follows:

Enter	15		0	-1,022	30,000
	N	I/Y	PV	PMT	FV
Solve for		9.0			

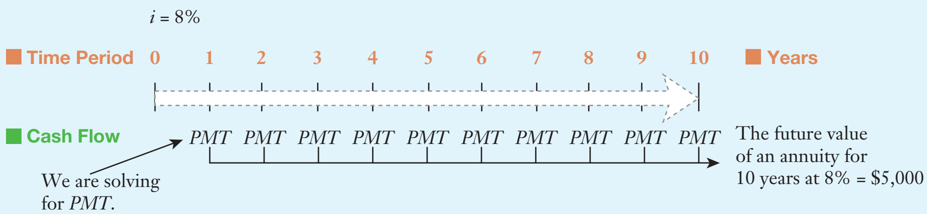
Checkpoint 6.1

Solving for an Ordinary Annuity Payment

How much must you deposit at the end of each year in a savings account earning 8 percent annual interest in order to accumulate $5,000 at the end of 10 years? Let's solve this problem using the mathematical formulas, a financial calculator, and an Excel spreadsheet.

STEP 1: Picture the problem

We can use a timeline to identify the annual payments earning 8 percent that must be made in order to accumulate $5,000 at the end of 10 years as follows:



STEP 2: Decide on a solution strategy

This is a future-value-of-an-annuity problem where we know the values for n, i, and FV and we are solving for PMT (PV is zero because there is no cash flow at Time Period 0). We'll use Equation (6–1c) to solve the problem.

STEP 3: Solution

Using the Mathematical Formulas.

Substituting these example values into Equation (6–1c), we find

$$\$5{,}000 = PMT\left[\frac{(1 + .08)^{10} - 1}{.08}\right]$$

$$\$5{,}000 = PMT(14.4866)$$

$$PMT = \$5{,}000 \div 14.4866 = \$345.15$$

Thus, you must deposit $345.15 in the bank at the end of each year for 10 years at 8 percent interest to accumulate $5,000.

Using a Financial Calculator.

Enter	10	8.0	0		5,000
	N	I/Y	PV	PMT	FV
Solve for				-345.15	

Using an Excel Spreadsheet.

= PMT(rate,nper,pv,fv) or, with values entered, = PMT(.08,10,0,5000)

STEP 4: Analyze

Notice that in a problem involving the future value of an ordinary annuity, the last payment actually occurs at the time the future value occurs. In this case, the last payment occurs at the end of Year 10, and the end of Year 10 is when you want the future value of the annuity to equal $5,000. In effect, the final payment does not have a chance to earn any interest.

STEP 5: Check yourself

If you can earn 12 percent on your investments and you would like to accumulate $100,000 for your newborn child's education at the end of 18 years, how much must you invest annually to reach your goal?

ANSWER: $1,793.73 at the end of each year.

Your Turn: For more practice, do related **Study Problems** 6–5, 6–17, 6–19, and 6–34 at the end of this chapter. >> **END Checkpoint 6.1**

Using an Excel Spreadsheet.

$$= \text{RATE(nper,pmt,pv,fv)} \text{ or, with values entered, } = \text{RATE}(15, -1022,0,30000)$$

Solving for the Number of Periods in an Ordinary Annuity

You may also want to calculate the *number of periods* it will take for an annuity to reach a certain future value. Just as with the calculation of the interest rate in an ordinary annuity, the easiest way to do this is with a financial calculator or a spreadsheet. For example, suppose you are investing $5,000 at the end of each year in an account that pays 7 percent. How long will it be before your account is worth $51,300?

Using a Financial Calculator. We can use a financial calculator to solve for n directly as follows:

Enter		7.0	0	-5,000	51,300
	N	I/Y	PV	PMT	FV
Solve for	8.0				

Thus, it will take eight years for end-of-year deposits of $5,000 every year to grow to $51,300.

Using an Excel Spreadsheet.

$$= \text{NPER(rate,pmt,pv,fv)} \text{ or, with values entered, } = \text{NPER}(7\%, -5000,0,51300)$$

The Present Value of an Ordinary Annuity

Let's say you just won a radio contest and the prize is $2,500. The only catch is that you are to receive the $2,500 in the form of five $500 payments at the end of each of the next five years. Alternatively, the radio station has offered to pay you a total of $2,000 today. Which alternative should you choose?

To make this decision, you will need to calculate the present value of the $500 annuity and compare it to the $2,000 lump sum. You can do this by discounting each of the individual future cash flows back to the present and then adding all the present values together. This can be a time-consuming task, particularly when the annuity lasts for several years. Nonetheless, it can be done. If you want to know what $500 received at the end of each of the next five years is worth today, assuming you can earn 6 percent interest on your investment, you simply substitute the appropriate values into Equation (5–2) :

$$\begin{aligned} PV &= \$500\left[\frac{1}{(1+.06)^1}\right] + \$500\left[\frac{1}{(1+.06)^2}\right] + \$500\left[\frac{1}{(1+.06)^3}\right] + \$500\left[\frac{1}{(1+.06)^4}\right] + \$500\left[\frac{1}{(1+.06)^5}\right] \\ &= \$500(0.94340) + \$500(0.89000) + \$500(0.83962) + \$500(0.79209) + \$500(0.74726) \\ &= \$2,106.18 \end{aligned}$$

Thus, the present value of this annuity is $2,106.18. As a result, you'd be better off taking the annuity rather than the $2,000 immediately. By examining the math and the timeline presented in Figure 6.2, you can see that the present values of the individual cash flows are simply summed. However, many times we will be faced with a situation where n, the number of cash flows in the annuity, is very large. For example, a 15-year mortgage involves 180 equal monthly payments, and a 30-year mortgage involves 360 equal monthly payments—that's just too many individual cash flows to work with. For this reason, we will want to use a financial calculator, Excel, or a mathematical shortcut. Let's examine a mathematical shortcut for valuing the present value of an annuity.

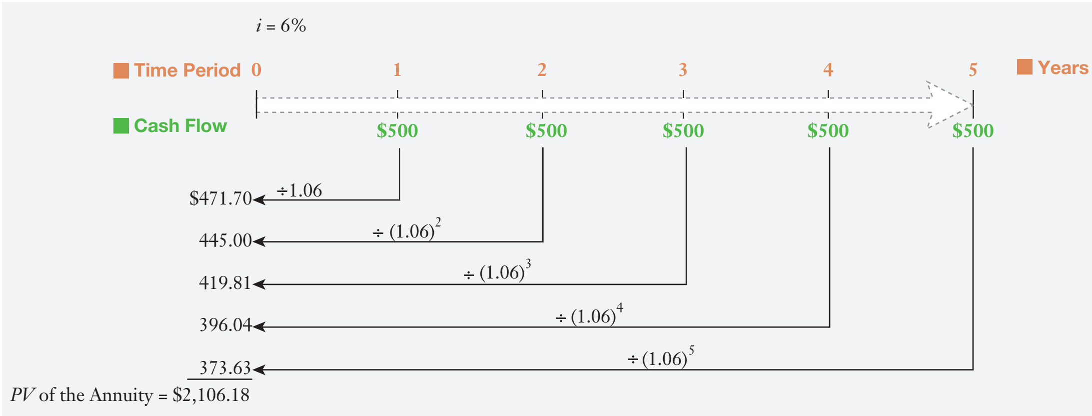
In this method for finding the present value of an annuity, we discount each cash flow separately and then add them up, as represented by the following equation:

$$PV = PMT\left[\left(\frac{1}{(1+i)^1}\right) + \left(\frac{1}{(1+i)^2}\right) + \cdots + \left(\frac{1}{(1+i)^n}\right)\right] \qquad \textbf{(6–2)}$$

Figure 6.2

Timeline of a Five-Year, $500 Annuity Discounted Back to the Present at 6 Percent

To find the present value of an annuity, discount each cash flow back to the present separately and then add them. In this example, we simply add up the present values of five future cash flows of $500 each to find a present value of $2,106.18.



>> END FIGURE 6.2

The term in brackets is commonly referred to as the **annuity present value interest factor**. We can simplify the present value interest factor for an annuity formula as follows:

$$\text{Annuity Present Value Interest Factor} = \frac{1 - \dfrac{1}{(1+i)^n}}{i} \tag{6–2a}$$

Thus, we can rewrite Equation (6–2) as follows:

$$PV = PMT\left[\frac{1 - \dfrac{1}{(1+i)^n}}{i}\right] \tag{6–2b}$$

Important Definitions and Concepts:

- *PV* is the present value of the annuity.
- *PMT* is the annuity payment deposited or received at the end of each period.
- *i* is the discount (or interest) rate on a per-period basis. For example, if annuity payments are received annually, *i* is expressed as an annual rate; if the payments are received monthly, it is a monthly rate.
- *n* is the number of periods for which the annuity will last. If the annuity payments are received annually, *n* is the number of years; if the payments are received monthly, it is the number of months.

Notice that the frequency of the payment—that is, whether payments are made on an annual, semiannual, or monthly basis—will play a role in determining the values of *n* and *i*. Moreover, it is important that *n* and *i* match; if periods are expressed in terms of the number of monthly payments, the interest rate must be expressed in terms of the interest rate per month. To find the present value of an annuity, all we need to do is multiply the annuity payment by the annuity present value interest factor.[2] Checkpoint 6.2 demonstrates the use of this formula, along with the other techniques for calculating the present value of an annuity.

[2] Related tables appear in Appendices B through E in MyLab Finance.

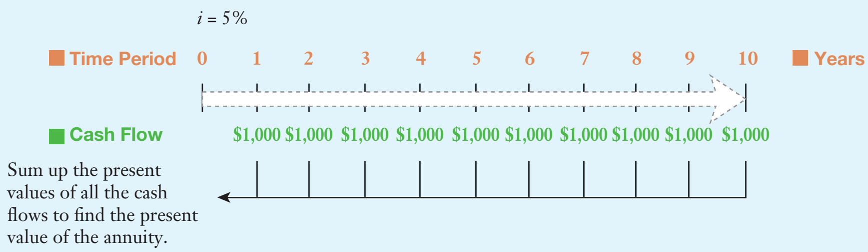
Checkpoint 6.2

The Present Value of an Ordinary Annuity

Your grandmother has offered to give you $1,000 per year for the next 10 years. What is the present value of this 10-year, $1,000 annuity discounted back to the present at 5 percent? Let's solve this using the mathematical formulas, a financial calculator, and an Excel spreadsheet.

STEP 1: Picture the problem

We can use a timeline to identify the cash flows from the investment as follows:



STEP 2: Decide on a solution strategy

In this case, we are trying to determine the present value of an annuity, and we know the dollar value that is received at the end of each year and the number of years the annuity lasts. We also know that the discount rate is 5 percent. We can use Equation (6–2b) to solve this problem.

STEP 3: Solution

Using the Mathematical Formulas. Substituting these example values into Equation (6–2b), we find that

$$PV = \$1{,}000\left[\frac{1 - \dfrac{1}{(1 + .05)^{10}}}{.05}\right] = \$1{,}000\,[(1 - .6139)/.05] = \$1{,}000\,(7.722) = \$7{,}721.73$$

Using a Financial Calculator.

Enter	10	5.0		1,000	0
	N	I/Y	PV	PMT	FV
Solve for			-7,721.73		

Using an Excel Spreadsheet.

= PV(rate,nper,pmt,fv) or, with values entered, = PV(0.05,10,1000,0)

STEP 4: Analyze

We will see this formula at work a bit later when we look at the value of a bond. When you buy a bond, you get the same amount of interest every year on either an annual or a semiannual basis, and then at maturity, you get the repayment of the bond's principal. Part of calculating the value of a bond involves calculating the present value of the bond's interest payments, which is an annuity.

STEP 5: Check yourself

What is the present value of an annuity of $10,000 to be received at the end of each year for 10 years, given a 10 percent discount rate?

ANSWER: $61,446.

Your Turn: For more practice, do related **Study Problems** 6–2, 6–4, 6–28, and 6–35 at the end of this chapter. >> **END Checkpoint 6.2**

Amortized Loans

An **amortized loan** is a loan paid off in equal payments—consequently, the loan payments are an annuity. *PV* can be thought of as the amount that has been borrowed, *n* is the number of periods the loan lasts, *i* is the interest rate per period, *FV* takes on a value of zero because the loan will be paid off after *n* periods, and *PMT* is the loan payment that is made. Generally, the payments are made monthly, but sometimes they are made yearly. Most mortgages and almost all car loans are amortized loans. Suppose you plan to get a $6,000 car loan at 15 percent annual interest with annual payments that you will pay off over four years. What will your annual payments be on this loan? Let's solve this using a financial calculator.

Enter	4	15.0	6,000		0
	N	I/Y	PV	PMT	FV
Solve for				-2,101.59	

The above calculation indicates that you would make annual payments of $2,101.59. Table 6.1 shows the breakdown of interest and principal over the life of the loan, which is commonly referred to as a **loan amortization schedule**.

As you can see, the interest payment declines each year as more of the principal is repaid and the amount owed declines. This is because a loan payment is made up of two parts: interest and principal. As part of each payment goes toward the principal, the size of the outstanding balance goes down. And as the size of the outstanding balance goes down, the amount of interest that is due in the next period declines. But because the size of each payment remains the same and the amount of the next payment that goes toward interest declines, the amount of the next payment that goes toward principal must increase. You can see this clearly in Table 6.1. If you look at Table 6.1, you'll see that the interest portion of the annuity (column 3) is calculated by multiplying the outstanding loan balance at the beginning of the year (column 1) by the interest rate of 15 percent. Thus, for the first year, the interest portion of the first year's payment is $\$6{,}000.00 \times .15 = \$900.00$; for Year 2, it is $\$4{,}798.41 \times .15 = \$719.76$; and so on. Of course, the amount that isn't the interest portion must be the principal portion. Thus, the repayment of the principal portion of the annuity is calculated by subtracting the interest portion of the annuity (column 3) from the annuity payment (column 2).

Amortized Loans with Monthly Payments

Many loans—for example, auto and home loans—require monthly payments. As we saw before, dealing with monthly, as opposed to yearly, payments is easy. All we do is multiply the number of years by *m,* the number of times compounding occurs during the year, to determine *n*, the number of periods. Then we divide the annual interest rate by *m* to find the interest rate per period.

Let's look at an example. You've just found the perfect home. However, in order to buy it, you'll need to take out a $150,000, 30 mortgage with monthly payments at an annual rate of 6 percent. What will your monthly mortgage payments be?

Table 6.1 Loan Amortization Schedule for a $6,000 Loan at 15% to Be Repaid in Four Years

Year	Amount Owed on the Principal at the Beginning of the Year (1)	Annuity Payment (2)	Interest Portion of the Annuity $= (1) \times 15\% = (3)$	Repayment of the Principal Portion of the Annuity $= (2) - (3) = (4)$	Outstanding Loan Balance at Year End, After the Annuity Payment $= (1) - (4) = (5)$
1	$6,000.00	$2,101.59	$900.00	$1,201.59	$4,798.41
2	4,798.41	2,101.59	719.76	1,381.83	3,416.58
3	3,416.58	2,101.59	512.49	1,589.10	1,827.48
4	1,827.48	2,101.59	274.12	1,827.48	0.00

Using the Mathematical Formulas. As we saw in the previous chapter in Equation (5–1b), in order to determine *n*, the number of periods, we multiply the number of years by *m,* where *m* is the number of times compounding occurs each year. To determine the interest rate per period, we divide the annual interest rate by *m*, where *m* is the number of times compounding occurs per year. Modifying Equation (6–2b) for non-annual compounding, we find

$$PV = PMT\left[\frac{1 - \dfrac{1}{(1 + \text{Annual Interest Rate}/m)^{(\text{Number of Years }(n)) \times m}}}{\text{Annual Interest Rate}/m}\right] \qquad \textbf{(6–2c)}$$

Substituting annual interest rate $= .06$, $n = 30$, $m = 12$, and $PV = \$150{,}000$ into Equation (6–2c), we get

$$\$150{,}000 = PMT\left[\frac{1 - \dfrac{1}{(1 + .06/12)^{30 \times 12}}}{.06/12}\right]$$

Notice that when you convert the annual rate of 6 percent to a monthly rate (by dividing it by 12), the monthly rate drops to 0.005, or 0.5 percent.

$$\$150{,}000 = PMT\left[\frac{1 - \dfrac{1}{(1 + .005)^{360}}}{.005}\right]$$

$$\$150{,}000 = PMT(166.7916144)$$

$$PMT = \$150{,}000/166.7916144 = \$899.33$$

Using a Financial Calculator. Because there are 360 monthly periods in 30 years, 360 is entered for N, and I/Y becomes 0.5 (annual interest rate of 6% divided by *m,* which is 12).

Enter	360	0.5	150,000		0
	N	I/Y	PV	PMT	FV
Solve for				-899.33	

Using an Excel Spreadsheet.

= PMT(rate,nper,pv,fv) or, with values entered, = PMT(0.005,360,150000,0)

Computing Your Outstanding Balance

Let's take a look at how you might use your understanding of annuities to calculate the outstanding balance on a home mortgage loan, which is equal to the present value of your future loan payments. Remember, when you solve for your payment, the final future value of the loan is zero because after your last payment is made, the loan is paid off. The present value of the loan represents how much you originally borrowed; that is, it is the initial outstanding loan balance. What all that means is that the *remaining outstanding balance on a loan must be equal to the present value of the remaining payments on that loan*. An example of this calculation is provided in Checkpoint 6.3.

Annuities Due

Thus far, we have looked only at ordinary annuities, annuities in which payments are made at equidistant points in time at the end of a period. Now we turn our attention to valuing an **annuity due**, an annuity in which all the cash flows occur at the beginning of a period. For example, rent payments on apartments are typically annuities due because the payment for the month's rent occurs at the beginning of the month. Fortunately, compounding annuities due and determining their future and present values are actually quite simple. Let's look at how this affects our compounding calculations.

Checkpoint 6.3

Determining the Outstanding Balance of a Loan

Let's say that exactly 10 years ago you took out a $200,000, 30-year mortgage with an annual interest rate of 9 percent and monthly payments of $1,609.25. But since you took out that loan, interest rates have dropped. You now have the opportunity to refinance your loan at an annual rate of 7 percent over 20 years. You need to know what the outstanding balance on your current loan is so you can take out a lower-interest-rate loan and pay it off. If you just made the 120th payment and have 240 payments remaining, what's your current loan balance?

STEP 1: Picture the problem

Because we are trying to determine how much you still owe on your loan, we need to determine the present value of your remaining payments. In this case, because we are dealing with a 30-year loan, with 240 remaining monthly payments, it's a bit difficult to draw a timeline that shows all the monthly cash flows. Still, we can mentally visualize the problem, which involves calculating the present value of 240 payments of $1,609.25 using a discount rate of 9%/12.

STEP 2: Decide on a solution strategy

Initially, you took out a $200,000, 30-year mortgage with an interest rate of 9 percent and monthly payments of $1,609.25. Because you have made 10 years' worth of payments—that's 120 monthly payments—there are only 240 payments left before your mortgage will be totally paid off. We know that the outstanding balance is the present value of all the future monthly payments. To find the present value of these future monthly payments, we'll use Equation (6–2c).

STEP 3: Solve

Using the Mathematical Formulas.

Using Equation (6–2c), we'll solve for the present value of the remaining monthly payments. To find n, we multiply the number of years left until the mortgage is paid off (20) times the number of months in a year (12). Thus, n becomes 240. The future value will be equal to zero because the loan will be fully paid off in 20 years. The payment will be $1,609.25, as given above. In effect, the present value of the payments you still need to make is how much you still owe:

$$PV = PMT\left[\frac{1 - \dfrac{1}{(1 + \text{Annual Rate of Interest}/m)^{(\text{Number of Years } (n)) \times m}}}{\text{Annual Rate of Interest}/m}\right] \qquad \textbf{(6–2c)}$$

where m = number of times compounding occurs per year.

Substituting annual interest rate = .09, n = 20, m = 12, and PMT = $1,609.25 into Equation (6–2c), we get

$$\begin{aligned} PV &= \$1{,}609.25\left[\frac{1 - \dfrac{1}{(1 + .09/12)^{20 \times 12}}}{.09/12}\right] \\ &= \$1{,}609.25(111.145) \\ &= \$178{,}860.02 \end{aligned}$$

Using a Financial Calculator.

Enter	240	9.0÷12		-1,609.25	0
	N	I/Y	PV	PMT	FV
Solve for			178,860.02		

Using an Excel Spreadsheet.

= PV(rate,nper,pmt,fv) or, with values entered, = PV((9/12)%, 240, −1609.25,0)

STEP 4: Analyze

To solve this problem, we began with our monthly payments. Then we determined what the present value was of the remaining payments—this is how much you still owe. Thus, after making 10 years of monthly payments on your $200,000 mortgage that originally had a maturity of 30 years and carries a 9 percent annual rate of interest with monthly payments of $1,609.25, you still owe $178,860.02.

The logic behind what was done here is that the amount you owe on a loan should be equal to the present value of the remaining loan payments. However, if interest rates drop and you decide to refinance your mortgage, you'd find that there are some real costs associated with refinancing that we haven't touched on here. For example, there is an application fee, an appraisal fee, legal and title search fees, an origination fee for processing the loan, and a prepayment penalty, all adding to the cost of refinancing. Once you decide on a mortgage refinancing lender, make sure that you get all of your mortgage refinancing terms in writing.

STEP 5: Check yourself

Let's assume you took out a $300,000, 30-year mortgage with an annual interest rate of 8 percent and monthly payments of $2,201.29. Because you have made 15 years' worth of payments (that's 180 monthly payments), there are another 180 payments left before your mortgage will be totally paid off. How much do you still owe on your mortgage?

ANSWER: $230,345.

Your Turn: For more practice, do related **Study Problem** 6–38 at the end of this chapter. >> **END Checkpoint 6.3**

Because an annuity due merely shifts the payments from the end of the period to the beginning of the period, we can calculate its future value by compounding the cash flows for one additional period. Specifically, the compound sum, or future value, of an annuity due is simply

$$FV_n(\text{Annuity Due}) = PMT\left[\frac{(1+i)^n - 1}{i}\right](1+i) \tag{6–3}$$

Recall that earlier we calculated the future value of a five-year ordinary annuity of $5,000 earning 6 percent interest to be $28,185.46. If we now assume this is a five-year annuity due, its future value increases from $28,185.46 to $29,876.59:

$$FV_n\,(\text{Annuity Due}) = \$5{,}000\left[\frac{(1+.06)^5 - 1}{.06}\right](1+.06) = \$28{,}185.46(1+.06) = \$29{,}876.59$$

The present value calculations also change for an annuity due. Because each cash flow is received one year earlier, its present value is discounted back for one less period. To determine the present value of an annuity due, we merely figure out what its present value would be if it were an ordinary annuity and multiply that value by $(1 + i)$. This, in effect, cancels out one year's discounting.[3]

$$PV(\text{Annuity Due}) = PMT\left[\frac{1 - \dfrac{1}{(1+i)^n}}{i}\right](1+i) \tag{6–4}$$

Let's go back to our radio contest example. Suppose the radio station offered you a five-year annuity due instead of an ordinary annuity. If you were given $500 at the beginning of each

[3] Within each of the Excel functions, you are given the option of identifying any cash flow as being at the beginning of a period. To solve for an annuity due in Excel, you simply change the value for "type" from 0 to 1. Recall that 0 is the default setting—the setting used to calculate an ordinary annuity. Consequently, if you don't designate a value for the variable "type," Excel will default to 0, or end-of-period payments. If you look at any of the Excel problems we have done so far, you'll notice that we have omitted entering a variable for "type," thus indicating that the cash flows occur at the end of each time period.

of those five years and were able to invest it at an interest rate of 6 percent, its value would increase from \$2,106.18 (the value of the ordinary annuity) to \$2,232.55:

$$PV(\text{Annuity Due}) = \$500\left[\frac{1 - \dfrac{1}{(1+.06)^5}}{.06}\right](1+.06)$$
$$= \$2{,}106.18(1+.06) = \$2{,}232.55$$

The result of all this is that both the future value and the present value of an annuity due are larger than those of an ordinary annuity because, in each case, all payments are received or paid earlier. Thus, when we *compound* an annuity due, the cash flows come at the beginning of the period rather than the end of the period. They are, in effect, invested one period earlier

Finance for Life

Saving for Retirement

If you understand Principle 1: **Money Has a Time Value**, you will have a better idea of why it's so important to begin saving for retirement as soon as possible. Putting off saving for just one year can have a big impact on the amount of money you have when you retire—in fact, it may reduce your retirement funds by over \$250,000.

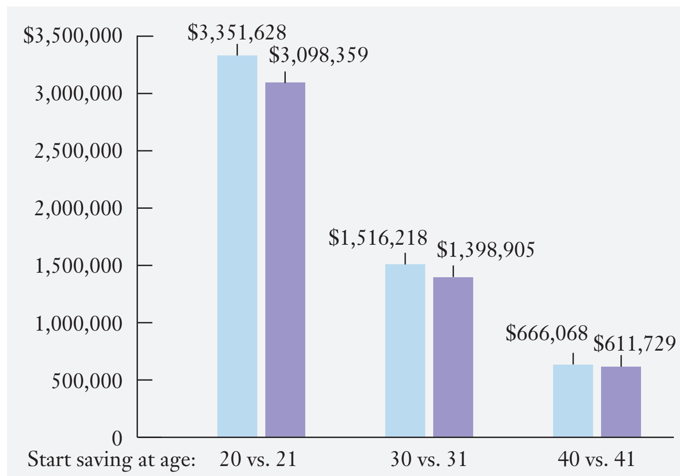
Individual retirement accounts, or IRAs, are personal retirement savings plans that have certain tax advantages. With a regular IRA, contributions are made on a before-tax basis. However, Roth IRA contributions are paid from earnings that have already been taxed. The difference is that after you retire and begin withdrawing money from a regular IRA, you have to pay taxes on your withdrawals. With a Roth IRA, you don't.

Figure 6.3 assumes that at a certain age you start contributing \$5,000 at the beginning of each year to a Roth IRA earning 8 percent interest per year and continue making these contributions until age 70. For example, if at age 20 you start contributing \$5,000 at the beginning of each year, you will have made 51 contributions by age 70, and you will end up with the following:

$$FV_n(\text{Annuity Due}) = PMT\left[\frac{(1+i)^n - 1}{i}\right](1+i)$$
$$= \$5{,}000\left[\frac{(1+.08)^{51} - 1}{.08}\right](1+.08) \qquad \textbf{(6–3)}$$
$$= \$3{,}103{,}359(1+.08) = \$3{,}351{,}628$$

Figure 6.3

Skipping Just One Year Can Cost You over a Quarter of a Million Dollars



>> **END FIGURE 6.3**

But what if you wait until you're 21 to start contributing? How much money will you end up with then? By waiting just one year longer to begin investing, you end up with \$253,269 less in the account:

$$FV(\text{Annuity Due}) = \$5{,}000\left[\frac{(1+.08)^{50} - 1}{.08}\right](1+.08)$$
$$= \$2{,}868{,}850(1+.08) = \$3{,}098{,}359$$

As you can see, instead of accumulating \$3,351,628, you accumulate only \$3,098,359.

In this example, you put \$5,000 in a Roth IRA each year, but if you had put in only \$2,500 each year, your total accumulation would be only half of what is listed in Figure 6.3. At present, the maximum you can put in your Roth IRA per year is \$5,000 unless you're over 50, in which case you can put even more in. Study Problem 6–3 looks at Roth IRAs.

Your Turn: See Study Problem 6.3.

and, as a result, grow to a larger future value. By contrast, when we *discount* an annuity due, the cash flows come at the beginning of the period, in effect coming one period earlier, so their present value is larger. Although annuities due are used with some frequency in accounting, their usage is less frequent in finance. Nonetheless, an understanding of annuities due can be powerful, as you can see in the feature *Finance for Life: Saving for Retirement*.

Before you move on to 6.2

Concept Check | 6.1

1. Define the term *annuity* as it relates to cash flows.
2. Distinguish between an ordinary annuity and an annuity due.
3. Describe the adjustments necessary when annuity payments occur on a monthly basis.
4. How would you determine how much you currently owe on an outstanding loan?

6.2 Perpetuities

A **perpetuity** is simply an annuity that continues forever or has no maturity. It is difficult to conceptualize such a cash flow stream that goes on forever. One such example, however, is the dividend stream on a share of preferred stock. In theory, this dividend stream will go on as long as the firm continues to pay dividends, so technically the dividends on a share of preferred stock form an infinite annuity, or perpetuity.

There are two basic types of perpetuities that we will encounter in our study of finance. The first is a **level perpetuity**, in which the payments are constant over time. The second is a **growing perpetuity**, in which the payments grow at a constant rate from period to period over time. Let's consider each in turn.

Calculating the Present Value of a Level Perpetuity

Determining the present value of a perpetuity is simple—you merely divide the constant flow, or payment, by the discount rate. For example, the present value of a $100 perpetuity discounted back to the present at 5 percent is $100/.05 = $2,000. The equation representing the present value of a level perpetuity is as follows:

$$PV_{\text{Level Perpetuity}} = \frac{PMT}{i} \tag{6–5}$$

Important Definitions and Concepts:

- $PV_{\text{Level Perpetuity}}$ = the present value of a level perpetuity.
- PMT = the constant dollar amount provided by the perpetuity.
- i = the interest (or discount) rate per period.

Calculating the Present Value of a Growing Perpetuity

Not all perpetuities have equal cash payments. In this text, we will encounter growing perpetuities when we discuss common stock valuation in Chapter 10. The type of growing perpetuity we will evaluate in Chapter 10 provides for the periodic cash flow to grow at a constant rate each period. For example, if the first payment at the end of Year 1 is $100 and the payments are assumed to grow at a rate of 5 percent per year, then the payment for Year 2 will be $100(1.05) = $105, and the payment for Year 3 will be $100(1.05)(1.05) = $110.25, and so forth.

We can calculate the present value of a growing perpetuity as follows:

$$PV_{\text{Growing Perpetuity}} = \frac{PMT_{\text{Period 1}}}{i - g} \tag{6–6}$$

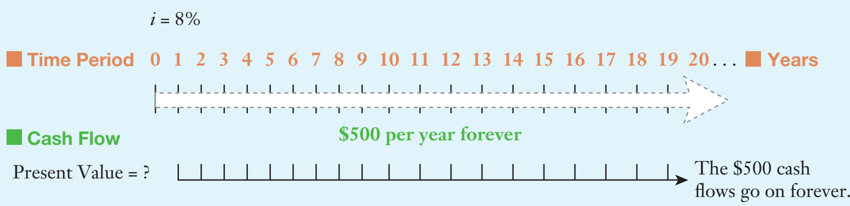
Checkpoint 6.4

The Present Value of a Level Perpetuity

What is the present value of a perpetuity of $500 paid annually discounted back to the present at 8 percent?

STEP 1: Picture the problem

With a perpetuity, a timeline doesn't have an ending point but goes on forever, with the same cash flow occurring period after period—in this case, year after year:



STEP 2: Decide on a solution strategy

Because calculating the present value of a perpetuity involves only simple division, we don't need to look at an Excel solution or any unique keystrokes with a financial calculator; instead, using Equation (6–5), we just divide the amount you received at the end of each period (forever) by the interest rate.

STEP 3: Solve

Substituting *PMT* = $500 and *i* = .08 into Equation (6–5), we find

$$pv_{\text{level perpetuity}} = \frac{\$500}{.08} = \$6{,}250$$

Thus, the present value of this perpetuity is $6,250.

STEP 4: Analyze

Notice there is no symbol for the future value of a perpetuity. This is because there isn't a future time period when things end; a perpetuity goes on indefinitely. So how much will this perpetuity be worth at the end of 2 years or 100 years? The answer is $6,250. That is because this perpetuity will always return $500—regardless of what the time period is, the present value of a perpetuity paying $500 at 8 percent is always $6,250.

STEP 5: Check yourself

What is the present value of a stream of payments equal to $90,000 paid annually and discounted back to the present at 9 percent?

ANSWER: $1,000,000.

Your Turn: For more practice, do related **Study Problem** 6–42 at the end of this chapter.

>> **END Checkpoint 6.4**

Important Definitions and Concepts:

- $PV_{\text{Growing Perpetuity}}$ = the present value of a growing perpetuity.
- $PMT_{\text{Period 1}}$ = the amount of the payment made at the end of the first period (e.g., this was $100 in the example used above).
- i = the rate of interest used to discount the growing perpetuity's cash flows.
- g = the rate of growth in the payment cash flows from period to period.

The growth rate, g, must be less than the rate of interest used to discount the cash flows, i. If g is greater than i, then the present value becomes infinitely large because the cash flows are growing at a faster rate than they are being discounted.

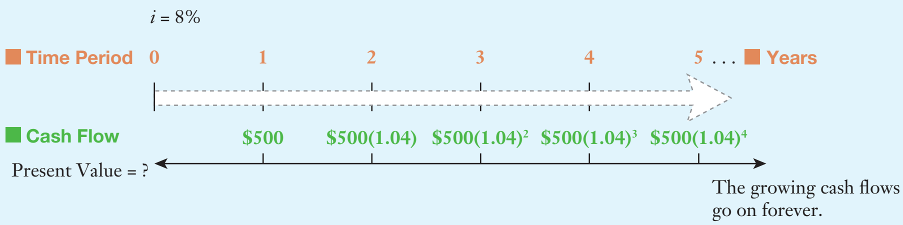
Checkpoint 6.5

The Present Value of a Growing Perpetuity

What is the present value of a perpetuity whose stream of cash flows pays \$500 at the end of Year 1 but grows at a rate of 4 percent per year indefinitely? The rate of interest used to discount the cash flows is 8 percent.

STEP 1: Picture the problem

With a growing perpetuity, a timeline doesn't have an ending point but goes on forever, with the cash flow growing at a constant rate period after period—in this case, year after year:



STEP 2: Decide on a solution strategy

Because calculating the present value of a growing perpetuity simply involves substituting into Equation (6–6), we don't need to look at an Excel solution or any unique keystrokes with a financial calculator. Instead, we just divide the amount received at the end of each period (forever) by the interest rate minus the growth rate.

STEP 3: Solve

Substituting $PMT_{\text{Period 1}} = \$500$, $g = .04$, and $i = .08$ into Equation (6–6), we find

$$PV_{\text{Growing Perpetuity}} = \frac{PMT_{\text{Period 1}}}{i - g} = \frac{\$500}{.08 - .04} = \$12{,}500$$

Thus, the present value of the growing perpetuity is \$12,500.

STEP 4: Analyze

Comparing the value of the \$500 level perpetuity in Checkpoint 6.4 to the \$500 perpetuity that grows at 4 percent per year, we see that adding growth to the cash flows has a dramatic effect on value. To see why this occurs, consider the Year 50 payment under both the level perpetuity and the growing perpetuity. For the level perpetuity, this payment is still \$500; however, for the growing perpetuity, the payment for Year 50 is

$$PMT_{\text{Year}_{50}} = \$500(1 + .04)^{50} = \$3{,}553.34$$

STEP 5: Check yourself

What is the present value of a stream of payments where the Year 1 payment is \$90,000 and the future payments grow at a rate of 5 percent per year? The interest rate used to discount the payments is 9 percent.

ANSWER: \$2,250,000.

Your Turn: For more practice, do related **Study Problem** 6–44 at the end of this chapter.

>> END Checkpoint 6.5

Before you move on to 6.3

Concept Check | 6.2

1. Define the term *perpetuity* as it relates to cash flows.
2. What is a growing perpetuity, and how is it calculated?

6.3 Complex Cash Flow Streams

Actual investment cash flows are often more complicated than the examples we have considered thus far. They frequently consist of multiple sets of annuities or different cash flow amounts mixed in with annuities. In general, they will involve spending money today in the hope of receiving more in the future, and once we bring all the future cash flows back to the present, they can be compared. For example, Marriott recently decided to build timeshare resorts in Dubai, United Arab Emirates. The resorts are close to Dubailand, a giant entertainment complex that is expected to open before 2020 and will be twice the size of the entire Disneyland and Disney World resorts put together.

The resorts' cash flows are a mixture of both positive and negative cash flows, as shown in Figure 6.4. The early cash flows are negative as Marriott begins construction on the various phases of the project and later become positive as the development makes money. Because of this mixture of positive and negative cash flows, we cannot use the annuities formulas that we described earlier. Instead, we calculate the present value of the investment project by summing the present values of all the individual cash flows.

Assuming a 6 percent discount rate, we can calculate the present value of all 10 years of cash flows by discounting each back to the present and then adding the positive flows and subtracting the negative ones. Note that the cash flows for Years 1 through 3 are different, so we will have to find their present values by discounting each cash flow back to the present. The present values of the payments (in millions of \$) received in Years 1 through 3 are \$471.70 $= \$500/(1 + .06)$, \$178.00 $= \$200/(1 + .06)^2$, and $-\ \$335.85 = -\ \$400/(1 + .06)^3$.

Next, we see that in Years 4 through 10 the cash flows correspond to an ordinary annuity of \$500 per year. Because these cash flows are all equal and are received annually, they are a seven-year annuity. The unique feature of the annuity is that the first cash flow comes at the end of Year 4. To find the present value of the seven-year annuity, we follow a two-step process:

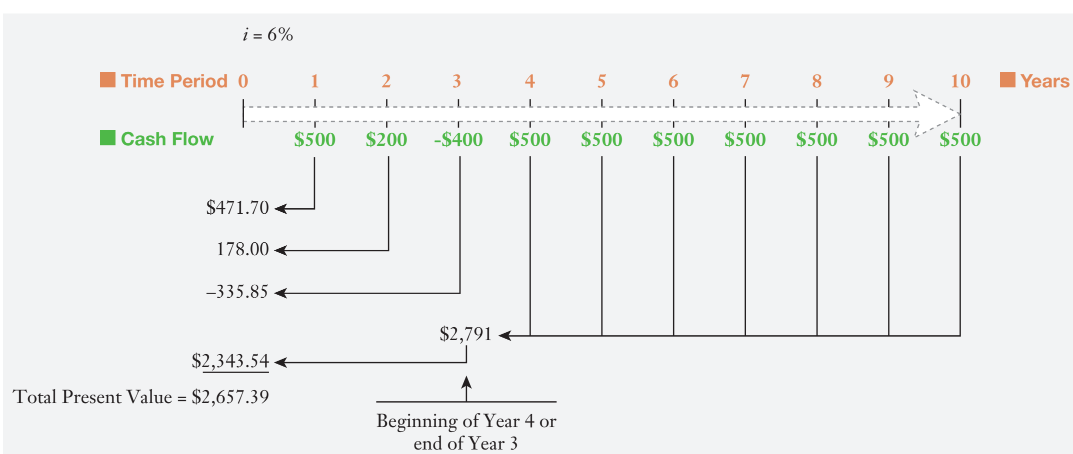
1. We consolidate the seven-year annuity into a single cash flow that is equal to its present value. In effect, we are consolidating the \$500 million payments that occur at the end of Years 4 through 10 into an equivalent single cash flow at the beginning of Year 4 (or the end of Period 3). Recall that we can find the present value of the annuity by multiplying the annual payment of \$500 by the annuity present value interest factor:

$$\frac{1 - \dfrac{1}{(1 + i)^n}}{i}$$

In Figure 6.4, we see that the present value of this annuity at the end of Year 3 is \$2,791 million.

Figure 6.4

Present Value of Single Cash Flows and an Annuity (\$ millions)



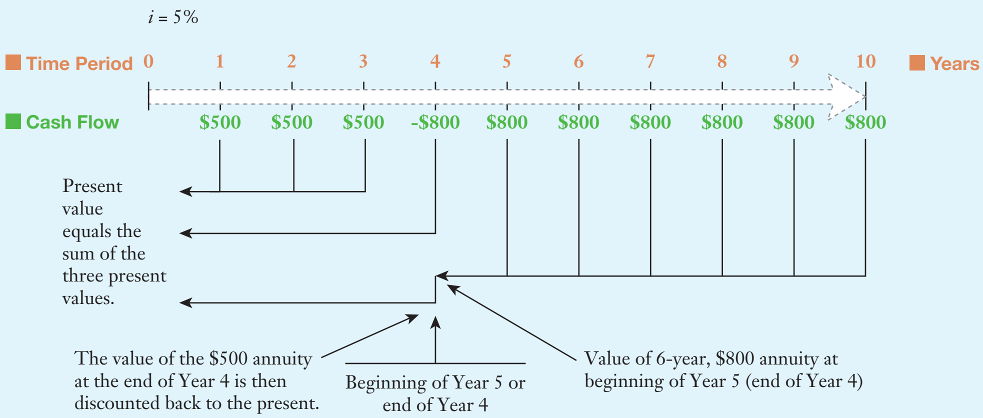
>> END FIGURE 6.4

Checkpoint 6.6

The Present Value of a Complex Cash Flow Stream

What is the present value of positive cash flows of $500 at the end of Years 1 through 3, a negative cash flow of $800 at the end of Year 4, and positive cash flows of $800 at the end of Years 5 through 10 if the appropriate discount rate is 5 percent?

STEP 1: Picture the problem



STEP 2: Decide on a solution strategy

This problem involves two annuities and the single (negative) cash flow. Once their present values are determined, they will be added together. The $500 annuity over Years 1 through 3 can be discounted directly to the present using Equation (6–2b), and the $800 cash outflow at the end of Year 4 can be discounted back to present using Equation (5–2) . Because the latter is an outflow, it will carry a negative sign and be subtracted from the present value of the inflows. To determine the present value of the six-year, $800 annuity over Years 5 through 10, we must first consolidate that annuity into an equivalent single cash flow at the beginning of Year 5, which is the same as the end of Year 4, using Equation (6–2b). We now have an equivalent single cash flow at the end of Year 4 that we can bring directly back to the present using Equation (5–2) . Once everything is in today's dollars, we simply add the values together.

STEP 3: Solve

Using the Mathematical Formulas. Here we have two annuities. One of them, an annuity of $500 over Years 1 through 3, can be discounted directly back to the present by multiplying it by the annuity present value interest factor:

$$\left[\frac{1-\dfrac{1}{(1+.05)^{3}}}{.05}\right]$$

for a value of $1,361.62. The second annuity, which is a six-year annuity of $800 per year over Years 5 through 10, must be discounted twice—once to find the value of the annuity at the beginning of Year 5, which is also the end of Year 4, and then again to bring that value back to the present. The value of that $800 annuity at the end of Year 4 is found by multiplying it by the annuity present value interest factor:

$$\left[\frac{1-\dfrac{1}{(1+.05)^{6}}}{.05}\right]$$

resulting in a value $4,060.55. In effect, we have now consolidated the annuity into an equivalent single cash flow at the end of Year 4. This equivalent single cash flow is then discounted back to the present by multiplying it by $1/(1.05)^4$, for a value of $3,340.62. Cash flows in the same time period can be added to and subtracted from each other, so to arrive at the total present value of this investment, we subtract the present value of the $800 cash outflow at the end of Year 4 (which is $658.16) from the sum of the present values of the two annuities

(6.3 CONTINUED >> ON NEXT PAGE)

($1,361.62 and $3,340.62). Thus, the present value of this series of cash flows is $4,044.08. Remember, once the cash flows from an investment have been brought back to the present, they can be combined by adding and subtracting to determine the project's total present value.

Using a Financial Calculator. Using a financial calculator, we can arrive at the same answer:

(a) The present value of the first annuity, Years 1 through 3 (give it a positive sign because it is an inflow) = **$1,361.62.**

	N	I/Y	PV	PMT	FV
Enter	3	5		500	0
Solve for			-1,361.62		

(b) The present value of the $800 cash outflow (give it a negative sign because it is an outflow) = **−$658.16.**

	N	I/Y	PV	PMT	FV
Enter	4	5		0	-800
Solve for			658.16		

(c) Part 1: The value at the end of Year 4 of the second annuity, Years 5 through 10 (give it a positive sign because it is an inflow) = **$4,060.55**.

	N	I/Y	PV	PMT	FV
Enter	6	5		800	0
Solve for			-4,060.55		

Part 2: The present value of the $4,060.55 that was calculated in Part 1 and is received at the end of Year 4 (give it a positive sign because it is an inflow) = **$3,340.62**.

	N	I/Y	PV	PMT	FV
Enter	4	5		0	4,060.55
Solve for			-3,340.62		

(d) Summing the present values, the total present value = **$4,044.08**.

Using an Excel Spreadsheet. Using Excel, the cash flows are brought back to the present using the = PV function, keeping in mind that inflows will take on positive signs and outflows negative signs.

STEP 4: Analyze

When cash flows from different time periods are expressed in the same time period's dollars, they can be added in the case of an inflow or subtracted in the case of an outflow to come up with a total value at some point in time. In fact, we will combine P Principle 1: **Money Has a Time Value** with P Principle 3: **Cash Flows Are the Source of Value** later in the book to value stocks, bonds, and investment proposals. The bottom line is that understanding the time value of money is a key to making good decisions.

STEP 5: Check yourself

What is the present value of cash flows of $300 at the end of Years 1 through 5, a cash flow of negative $600 at the end of Year 6, and cash flows of $800 at the end of Years 7 through 10 if the appropriate discount rate is 10 percent?

ANSWER: $2,230.

Your Turn: For more practice, do related **Study Problem** 6–48 at the end of this chapter.

>> **END** Checkpoint 6.6

2. We discount the \$2,791 million present value of the annuity cash flows back three years to the present, which is the beginning of Year 1. The present value of this sum (in millions of \$), then, is $\$2{,}343.54 = \$2{,}791/(1 + .06)^3$.

Finally, we can calculate the present value of the complex set of future cash flows by adding up the individual present values of the future cash flows. The result is a present value of \$2,657.39 million. We can then compare the present value of all the future cash flows with what the project costs. It should now be apparent that drawing out a timeline is a critical first step when trying to solve any complex problem involving the time value of money.

Before you begin end-of-chapter material

Concept Check | 6.3

1. When are cash flows comparable—that is, when can they be added together or subtracted from each other?
2. Why would you want to be able to compare cash flows that occur in different time periods with each other?

Applying the Principles of Finance to Chapter 6

P Principle 1: **Money Has a Time Value** This chapter continues our study of the time value of money—a dollar received today, other things being the same, is worth more than a dollar received a year from now. In this chapter, we expand on what we learned in the previous chapter by applying the time value of money to annuities, perpetuities, and complex cash flows.

P Principle 3: **Cash Flows Are the Source of Value** In this chapter, we introduced the idea that we will use **Principle 1** in combination with **Principle 3** to value stocks, bonds, and investment proposals.

Chapter Summaries

6.1 Distinguish between an ordinary annuity and an annuity due, and calculate the present and future values of each. (pgs. 192–205)

SUMMARY: An annuity is a series of equal dollar payments, where the periods between the payments are of equal length, such as monthly or annually. An ordinary annuity involves cash payments made at the end of each period, whereas an annuity due involves payments made at the beginning of each period. Appendices B through E in MyLab Finance contain tables with future value interest factors, present value interest factors, annuity future value interest factors, and annuity present value interest factors for various combinations of i and n.

KEY TERMS

Amortized loan, page 200 A loan that is paid off in equal periodic payments.

Annuity, page 192 A series of equal dollar payments for a specified period of time.

Annuity due, page 201 A series of equal dollar payments for a specified period of time in which the payments occur at the beginning of each period.

Annuity future value interest factor, page 194 The value $\left[\frac{(1+i)^n - 1}{i}\right]$, which is used as a multiplier to calculate the future value of an annuity.

Annuity present value interest factor, page 198 The value $\left[\frac{1 - \frac{1}{(1+i)^n}}{i}\right]$, which is used as a multiplier to calculate the present value of an annuity.

Loan amortization schedule, page 200 A breakdown of the interest and principal payments on an amortized loan.

Ordinary annuity, page 192 A series of equal dollar payments for a specified period of time in which the payments occur at the end of each period.

KEY EQUATIONS

$$FV_n = PMT\left[\frac{(1+i)^n - 1}{i}\right] \qquad \textbf{(6–1c)}$$

$$PV = PMT\left[\frac{1 - \frac{1}{(1+i)^n}}{i}\right] \qquad \textbf{(6–2b)}$$

$$PV = PMT\left[\frac{1 - \frac{1}{(1 + \text{Annual Interest Rate}/m)^{(\text{Number of Years }(n)) \times m}}}{\text{Annual Interest Rate}/m}\right] \qquad \textbf{(6–2c)}$$

$$FV_n = PMT\left[\frac{(1 + \text{Annual Interest Rate}/m)^{(\text{Number of Years }(n)) \times m} - 1}{\text{Annual Interest Rate}/m}\right]$$

$$FV_n(\text{Annuity Due}) = PMT\left[\frac{(1+i)^n - 1}{i}\right](1+i) \qquad \textbf{(6–3)}$$

Concept Check | 6.1

1. Define the term *annuity* as it relates to cash flows.
2. Distinguish between an ordinary annuity and an annuity due.
3. Describe the adjustments necessary when annuity payments occur on a monthly basis.
4. How would you determine how much you currently owe on an outstanding loan?

$$PV(\text{Annuity Due}) = PMT\left[\frac{1 - \dfrac{1}{(1+i)^n}}{i}\right](1+i) \qquad \textbf{(6–4)}$$

6.2 Calculate the present value of a level perpetuity and a growing perpetuity.

(pgs. 205–207)

SUMMARY: A perpetuity is an annuity that continues forever. That is, every period it pays the same dollar amount. With a growing perpetuity, rather than receiving the same amount each period, the periodic payment increases at a constant rate every period.

KEY TERMS

Growing perpetuity, page 205 An annuity in which the payments grow at a constant rate from period to period over an infinite life.

Level perpetuity, page 205 An annuity with a constant level of payments over an infinite life.

Perpetuity, page 205 An annuity with an infinite life.

KEY EQUATIONS

$$PV_{\text{Level Perpetuity}} = \frac{PMT}{i} \qquad \textbf{(6–5)}$$

$$PV_{\text{Growing Perpetuity}} = \frac{PMT_{\text{Period 1}}}{i - g} \qquad \textbf{(6–6)}$$

Concept Check | 6.2

1. Define the term *perpetuity* as it relates to cash flows.
2. What is a growing perpetuity, and how is it calculated?

6.3 Calculate the present and future values of complex cash flow streams.

(pgs. 208–211)

SUMMARY: Understanding how to make cash flows that occur in different time periods comparable is essential to understanding finance. All time value formulas presented in this chapter and in the previous chapter stem from the single compounding formula $FV_n = PV(1+i)^n$. Many times projects or investments involve combinations of cash flows—for example, discounting single flows, compounding annuities, and discounting annuities—and we find the present value of these complex combinations by calculating the present value of each cash flow and them adding the present values together.

Concept Check | 6.3

1. When are cash flows comparable—that is, when can they be added together or subtracted from each other?
2. Why would you want to be able to compare cash flows that occur in different time periods with each other?

Study Questions

6–1. What is an annuity? Give some examples of annuities.

6–2. How do you calculate the future value of an annuity?

6–3. What is the relationship between the present value interest factor (from Chapter 5) and the annuity present value interest factor (from Equation [6–2])?

6–4. Assume you bought a home and took out a 30-year mortgage on it 10 years ago. How would you determine how much principal on your mortgage you still have to pay off?

6–5. Distinguish between an ordinary annuity and an annuity due.

6–6. What is a level perpetuity? A growing perpetuity?

6–7. How do you calculate the present value of an annuity? A perpetuity? A growing perpetuity?

6–8. With an uneven stream of future cash flows, the present value is determined by discounting all of the cash flows back to the present and then adding the present values. Is there ever a time when you can treat some of the cash flows as annuities and apply the annuity techniques you learned in this chapter?

Study Problems

MyLab Finance
Go to **www.myfinancelab.com** to complete these exercises online and get instant feedback.

Annuities

6–1. **(Calculating the future value of an ordinary annuity)** Calculate the future value of each of the following streams of payments.

a. £430 a year for 12 years compounded annually at 6 percent.
b. €56 a year for 8 years compounded annually at 8 percent.
c. $75 a year for 5 years compounded annually at 3 percent.
d. £120 a year for 3 years compounded annually at 10 percent.

6–2. **(Calculating the present value of an ordinary annuity)** **(Related to Checkpoint 6.2 on page 199)** Calculate the present value of the following annuities.

a. £350 a year for 12 years discounted back to the present at 7 percent.
b. €260 a year for 5 years discounted back to the present at 8 percent.
c. $3,000 a year for 8 years discounted back to the present at 6 percent.
d. £60 a year for 3 years discounted back to the present at 10 percent.

6–3. **(Calculating the future value of an ordinary annuity)** **(Related to *Finance for Life: Saving for Retirement* on page 204)** Mike, 30, has started a small business and wants to start saving for his retirement when he turns 65. He has decided to invest £6,000 at the end of every year in a pension plan that earns him 6 percent compounded annually. How much will Mike get when he retires?

Mike then decided to keep reinvesting the profits in his business for a further five years and start saving once he is 35 years old. How much will Mike get if he starts saving the same amount and investing in the same plan as before?

6–4. **(Calculating the present value of an ordinary annuity)** **(Related to Checkpoint 6.2 on page 199)** Nicki Johnson, a sophomore mechanical engineering student, received a call from an insurance agent who believes that Nicki is an older woman who is ready to retire from teaching. He talks to her about several annuities that she could buy that would guarantee her a fixed annual income. The annuities are as follows:

Annuity	Purchase Price of the Annuity	Amount of Money Received per Year	Duration of the Annuity (years)
A	$50,000	$8,500	12
B	$60,000	$7,000	25
C	$70,000	$8,000	20

Nicki could earn 11 percent on her money by placing it in a savings account. Alternatively, she could place it in any of the above annuities. Which annuities in the table above, if any, will earn Nicki a higher return than investing in the savings account earning 11 percent?

6–5. **(Calculating annuity payments)** **(Related to Checkpoint 6.1 on page 196)** James Harrison bought a house for £180,000. He paid £20,000 upfront from his savings and took a mortgage to pay the rest for 25 years. The mortgage was to be paid in 25 equal annual installments that included both principal and interest. This mortgage charged 6 percent compound interest on unpaid balance. What will his annual installments be?

6–6. **(Calculating annuity payments)** Donna Langley bought a new luxury car for €50,000. She made a down payment of €5,000 and agreed to pay the rest over the next eight years in eight equal annual payments that include both principal and 12 percent compound interest on unpaid balance. What will these equal payments be?

6–7. **(Calculating annuity payments)** Jamie Oliver has just graduated and found a job. He had taken a student loan of $40,000 to pay for his university. He has decided to pay off this loan by making equal annual payments over the next 20 years. If the loan charges him compound interest at 8 percent per annum, what is the annual amount, including interest and principal, that he needs to repay?

6–8. **(Calculating annuity payments)** Satya Kumar wishes to have £30,000 at the end of 20 years to pay for his daughter's pilot training. He plans to do so by depositing an equal amount every year in an account that pays 7 percent compounded annually. How much does he need to deposit every year to reach his goal?

6–9. **(Calculating annuity payments)** Kelly Yeo plans to buy her own apartment in Singapore in 10 years' time when she retires. She has identified a location where house prices average at $300,000 at present. She also expects a rise in the house price by 5 percent every year. She plans to have the funds to buy the apartment at the time of her retirement by depositing an equal amount of money every year in a bank account that pays 9 percent per annum compound interest. How much does she need to deposit every year in this account to reach her goal at the time of retirement?

6–10. **(Calculating annuity payments)** The Aggarwal Corporation needs to save $10 million to retire a $10 million mortgage that matures in 10 years. To retire this mortgage, the company plans to put a fixed amount into an account at the end of each year for 10 years. The Aggarwal Corporation expects to earn 9 percent annually on the money in this account. What equal annual contribution must the firm make to this account to accumulate the $10 million by the end of 10 years?

6–11. **(Calculating annuity payments)** Sheef Metal PLC's factory is on leasehold land. The lease will end in 15 years' time from now. The directors have decided to buy the title rights from the leaseholder at the end of the lease term, and this is expected to cost £500,000 at the time of the lease's expiry. The firm plans to put a fixed amount into an account at the end of every year for the next 15 years to buy the title rights. If the account earns 8 percent every year compounded annually, how much will be the fixed amount that Sheef Metal PLC needs to put in the account?

6–12. **(Calculating the future value of an annuity)** Upon graduating from college 35 years ago, Dr. Nick Riviera was already planning for his retirement. Since then, he has made $300 deposits into a retirement fund on a quarterly basis. Nick has just completed his final payment and is at last ready to retire. His retirement fund has earned 9 percent compounded quarterly.

a. How much has Nick accumulated in his retirement account?
b. In addition to this, 15 years ago Nick received an inheritance check for $20,000 from his beloved uncle. He decided to deposit the entire amount into his retirement fund. What is his current balance in the fund?

6–13. **(Calculating the number of annuity periods)** Greer is borrowing €300,000 to buy a house in Amsterdam at 5 percent per annum compounded monthly. How long will it take to pay off the amount if she makes monthly payments of €1,600?

6–14. **(Calculating the number of annuity periods)** Alex Karev has taken out a $200,000 loan with an annual rate of 8 percent compounded monthly to pay off hospital bills from his wife Izzy's illness. If the most Alex can afford to pay is $1,500 per month, how long will it take for him to pay off the loan? How long will it take for him to pay off the loan if he can pay $2,000 per month?

6–15. **(Calculating the present value of annuity)** What should be the present value of an eight-year annuity payment that pays $1,700 annually given an applicable discount rate of 5 percent?

6–16. **(Calculating an annuity's interest rate)** Your folks would like some advice from you. An insurance agent just called and offered them the opportunity to purchase an annuity for $21,074.25 that will pay them $3,000 per year for 20 years. They don't have the slightest idea what return they would be making on their investment of $21,074.25. What rate of return would they be earning?

6–17. **(Calculating annuity payments)** **(Related to Checkpoint 6.1 on page 196)** On December 31, Beth Klemkosky bought a yacht for $50,000. She paid $10,000 down and agreed to pay the balance in 10 equal annual installments that include both principal and 10 percent interest on the declining balance. How big will the annual payments be?

6–18. **(Calculating an annuity's interest rate)** Rebecca Smyth has seen a deal for a new car worth £20,000 that she can take home now and start paying for only at the end of the year. She will have to make an annual payment of £6,000 every year for next four years. What will be the applicable rate of interest if she takes up the offer?

6–19. **(Calculating annuity payments)** **(Related to Checkpoint 6.1 on page 196)** Xiang Lu has started a business in Hong Kong and has borrowed $60,000 from a bank at 10 percent compounded annually. This loan will be repaid in equal installments at the end of each year over the next seven years. How much will each annual installment be?

6–20. **(Calculating annuity payments)** Mike and Bethan plan to buy their dream house in six years' time. They expect to need £30,000 to do so at that time. They are saving in an account that pays 10 percent per annum. How much do they have to deposit every year to have the required amount when the final deposit is made?

6–21. **(Calculating the number of annuity periods)** Rajesh has just bought a large flat screen TV for € 4,000 on hire purchase. He has agreed to pay 16 percent per annum compounded monthly interest on this amount. He has agreed to a monthly payment of €200. How long will he take to repay the loan? How much will he pay in interest on this loan?

6–22. **(Solving a comprehensive problem)** You would like to have $75,000 in 15 years. To accumulate this amount, you plan to deposit an equal sum in the bank each year that will earn 8 percent interest compounded annually. Your first payment will be made at the end of the year.

a. How much must you deposit annually to accumulate this amount?
b. If you decide to make a large lump-sum deposit today instead of the annual deposits, how large should the lump-sum deposit be? (Assume you can earn 8 percent on this deposit.)
c. At the end of five years, you will receive $20,000 and deposit it in the bank in an effort to reach your goal of $75,000 at the end of 15 years. In addition to the deposit, how large must your equal annual deposits for all 15 years be to reach your goal? (Again, assume you can earn 8 percent on this deposit.)

6–23. **(Calculating annuity payments)** Shuting has given herself five years to save $40,000 to buy a commercial property in Hong Kong for her business. She plans to make annual deposits in an account paying 10 percent per annum. If she makes her first deposit at the end of this year, what amount does she need to deposit every year to get the required amount of $40,000 at the end of a five-year period?

6–24. **(Calculating the future value of an annuity and annuity payments)** You are trying to plan for retirement in 10 years, and, currently, you have $150,000 in a savings account and $250,000 in stocks. In addition, you plan to deposit $8,000 per year into your savings account at the end of each of the next five years and then $10,000 per year at the end of each year for the final five years until you retire.

a. Assuming your savings account returns 8 percent compounded annually and your investment in stocks returns 12 percent compounded annually, how much will you have at the end of 10 years?
b. If you expect to live for 20 years after you retire and at retirement you deposit all of your savings in a bank account paying 11 percent, how much can you withdraw each year after you retire (making 20 equal withdrawals beginning one year after you retire) so that you end up with a zero balance at death?

6–25. **(Calculating annuity payments)** On December 31, Son-Nan Chen borrowed $100,000, agreeing to repay this sum in 20 equal annual installments that include both principal and 15 percent interest on the declining balance. How large will the annual payments be?

6–26. **(Calculating annuity payments)** Amrit Kolar bought a new house by borrowing £300,000 on a mortgage at 6 percent per annum repayable over 30 years in equal annual payments. How large will his annual payments be?

6–27. **(Calculating components of annuity payments)** Omar Khalid has started a new factory and bought a commercial building for $160,000 with an 8 percent mortgage to be paid over 20 years, calling for payment semi-annually. What will be the semiannual payment? What will the interest and principal components be in the first two installments of the first year?

6–28. **(Calculating the present value of annuity payments)** **(Related to Checkpoint 6.2 on page 199)** The state lottery's million-dollar payout provides for $1 million to be paid over the course of 19 years in amounts of $50,000. The first $50,000 payment is made immediately, and the 19 remaining $50,000 payments occur at the end of each of the next 19 years. If 10 percent is the discount rate, what is the present value of this stream of cash flows? If 20 percent is the discount rate, what is the present value of the cash flows?

6–29. **(Calculating the future value of an annuity)** Find the future value of an annuity that pays €8,000 a year for 10 years at 6 percent compounded annually. What will be the future value if it was compounded at 10 percent?

6–30. **(Calculating the present value of an annuity due)** What will be the present value of an annuity due of £800 a year for 12 years, discounted back to the present at an annual rate of 5 percent? What will be the present value of this annuity if the discount rate is 8 percent?

6–31. **(Calculating the present value of annuity)** You have agreed to invest in a new business scheme that has promised to pay £2,000 every year starting at the end of Year 5 from now, assuming it earns 9 percent per annum. This will continue to pay the same amount each year for 10 years once started. What will the present value of this business opportunity be?

6–32. **(Calculating the components of an annuity payment)** You have just bought a house for €270,000 by taking a 20-year mortgage for the same amount at 8 percent per annum payable in monthly installments. What will your monthly payments be? Use a spreadsheet to calculate your answer. Now calculate the amounts in the 50th monthly payment that goes toward interest and principal, respectively.

6–33. **(Solving a comprehensive problem)** Over the past few years, Microsoft founder Bill Gates's net worth has fluctuated between $20 and $130 billion. In early 2006, it was about $26 billion—after he reduced his stake in Microsoft from 21 percent to around 14 percent by moving billions into his charitable foundation. Let's see what Bill Gates can do with his money in the following problems.

a. Manhattan's native tribe sold Manhattan Island to Peter Minuit for $24 in 1626. Now, 390 years later in 2016, Bill Gates wants to buy the island from the "current natives." How much will Bill have to pay for Manhattan if the "current natives" want a 6 percent annual return on the original $24 purchase price?

b. Bill Gates decides to pass on Manhattan and instead plans to buy the city of Seattle, Washington, for $50 billion in 10 years. How much will Bill have to invest today at 10 percent compounded annually in order to purchase Seattle in 10 years?

c. Now assume Bill Gates wants to invest only about 17 percent of his net worth, which stood at around $76 billion in 2016, or $13 billion, in order to buy Seattle for $50 billion in 10 years. What annual rate of return will he have to earn in order to complete his purchase in 10 years?

d. Instead of buying and running large cities, Bill Gates is considering quitting the rigors of the business world and retiring to work on his golf game. To fund his retirement, Bill wants to invest his $20 billion fortune in safe investments with an expected annual rate of return of 7 percent. He also wants to make 40 equal annual withdrawals from this retirement fund beginning a year from today, running his retirement fund to $0 at the end of 40 years. How much can his annual withdrawal be in this case?

6–34. **(Calculating annuity payments)** **(Related to Checkpoint 6.1 on page 196)** Sheryl Williams wants to have a million dollars when she retires, 40 years from now. She is planning to do this by depositing an equal amount at the end of every year for the next 40 years. If her tax-free savings account pays her 9 percent per annum, how much does she need to deposit every year?

6–35. **(Calculating the present value of annuity)** **(Related to Checkpoint 6.2 on page 199)** Xiang Lu received €80,000 four years ago as an inheritance, which he immediately deposited in an account paying him 5 percent every year. Now he has started to put in €3,000 every year in the same account, starting now. How much money will he have at the end of 25 years?

6–36. **(Calculating annuity payments)** Professor Finance is thinking about trading cars. She estimates she will still have to borrow $25,000 to pay for her new car. How large will Prof. Finance's monthly car loan payment be if she can get a five-year (60 equal monthly payments) car loan from the VTech Credit Union at 6.2 percent APR?

6–37. **(Calculating annuity payments)** Ford Motor Company's current incentives include a choice between 4.9 percent APR financing for 60 months and $1,000 cash back on a Mustang. Let's assume Suzie Student wants to buy the premium Mustang convertible, which costs $25,000, and she has no down payment other than the cash back from Ford. If she chooses $1,000 cash back, Suzie can borrow from the VTech Credit Union at 6.9 percent APR for 60 months (Suzie's credit isn't as good as that of Prof. Finance). What will Suzie Student's monthly payment be under each option? Which option should she choose?

6–38. **(Determining the outstanding balance of a loan)** **(Related to Checkpoint 6.3 on page 202)** Mrs. Khan took a mortgage of £140,000 at 5 percent per annum 10 years ago for a period of 25 years. She pays a monthly payment of £818.43. What is the outstanding balance on her current loan if she has just paid her 120th payment?

6–39. **(Calculating annuity payments)** Calvin Johnson has a $5,000 debt balance on his Visa card that charges 12.9 percent APR compounded monthly. Let's assume Calvin's only needed to make a minimum monthly payment of 3 percent of his debt balance, which is $150. How many months (round up) will it take Calvin Johnson to pay off his credit card if he pays $150 at the end of each month? Now let's assume that the minimum monthly payment on credit cards rises to 4 percent. If Calvin makes monthly payments of $200 at the end of each month, how long will it take to pay off his credit card?

6–40. **(Calculating the future value of an annuity)** Let's say you deposited $160,000 in a 529 plan (a tax-advantaged college savings plan), hoping to have $420,000 available 12 years later when your first child starts college. However, you didn't invest very well, and two years later the account's balance dropped to $140,000. Let's look at what you need to do to get the college savings plan back on track.

a. What was the original annual rate of return needed to reach your goal when you started the fund two years ago?

b. With only $140,000 in the fund and 10 years remaining until your first child starts college, what annual rate of return will the fund have to make to reach your $420,000 goal if you add nothing to the account?

c. Shocked by your experience of the past two years, you feel the college fund has invested too much in stocks, and you want a low-risk fund in order to ensure you have the necessary $420,000 in 10 years. You are willing to make end-of-the-month deposits to the fund as well. You find you can get a fund that promises to pay a guaranteed annual return of 6 percent that is compounded monthly. You decide to transfer the $140,000 to this new fund and make the necessary monthly deposits. How large of a monthly deposit must you make in this new fund each month?

d. After seeing how large the monthly deposit has to be (in part c of this problem), you decide to invest the $140,000 today and $500 at the end of each month for the next 10 years in a fund consisting of 50 percent stock and 50 percent bonds and to hope for the best. What APR will the fund have to earn in order to reach your $420,000 goal?

6–41. **(Calculating the future value of an annuity)** Selma and Patty Bouvier, twins who work at the Springfield DMV, have decided to save for retirement, which is 35 years away. They will both receive an 8 percent annual return on their investment over the next 35 years. Selma invests $2,000 at the end of each year *only* for the first 10 years of the 35-year period—for a total of $20,000 saved. Patty doesn't start saving for 10 years and then saves $2,000 at the end of each year for the remaining 25 years—for a total of $50,000 saved. How much will each of them have when they retire?

Perpetuities

6–42. **(Calculating the present value of a perpetuity)** **(Related to Checkpoint 6.4 on page 206)** What is the present value of the following?

a. A $300 perpetuity discounted back to the present at 8 percent
b. A $1,000 perpetuity discounted back to the present at 12 percent
c. A $100 perpetuity discounted back to the present at 9 percent
d. A $95 perpetuity discounted back to the present at 5 percent

6–43. **(Calculating the present value of a perpetuity)** What will be the present value of a perpetual payment of £400 per year if the applicable discount rate is 6 percent? What will be its value if the discount rate is changed to 3 percent?

6–44. **(Calculating the present value of a growing perpetuity)** **(Related to Checkpoint 6.5 on page 207)** A perpetuity pays $1,000 at the end of Year 1, and the annual cash flows grow at a rate of 4 percent per year indefinitely. What is the present value if the appropriate discount rate is 8 percent? If the appropriate discount rate is 6 percent?

6–45. **(Calculating the present value of a growing perpetuity)** A pension plan pays €30,000 at the end of Year 1 and then grows at the rate of 3 percent per year indefinitely. What is the present value if the rate of interest to discount the cash flow is 7 percent?

6–46. **(Calculating the present value of a growing perpetuity)** Jonathan Lee wants to start an annual bursary of €50,000 for his alma mater in Berlin. The first payment will occur at the end of this year and it will then grow by 5 percent each year to cover for inflation. If the bank account pays 8 percent per annum, how much does Jonathan needs to invest now to support this bursary forever?

6–47. **(Calculating the present value of a negatively growing perpetuity)** Your firm has taken cost-saving measures that will provide a benefit of $10,000 in the first year. These cost savings will decrease each year at a rate of 3 percent forever. If the appropriate interest rate is 6 percent, what is the present value of these savings?

Complex Cash Flow Streams

6–48. **(Calculating the present value of annuities and complex cash flows)** **(Related to Checkpoint 6.6 on page 209)** You are given three investment alternatives to analyze. The cash flows from these three investments are as follows:

End of Year	A	B	C
1	$10,000		$10,000
2	10,000		
3	10,000		
4	10,000		
5	10,000	$10,000	
6		10,000	50,000
7		10,000	
8		10,000	
9		10,000	
10		10,000	10,000

Assuming a 20 percent discount rate, find the present value of each investment.

6–49. **(Calculating the present value of annuities and complex cash flows)** You are given three investment alternatives to analyze. The cash flows from these three investments are as follows:

End of Year	A	B	C
1	$15,000		$20,000
2	15,000		
3	15,000		
4	15,000		
5	15,000	$15,000	
6		15,000	60,000
7		15,000	
8		15,000	
9		15,000	
10		15,000	20,000

Assuming a 20 percent discount rate, find the present value of each investment.

6–50. **(Calculating the present value of an uneven stream of payments)** You are given three investment alternatives to analyze. The cash flows from these three investments are as follows:

End of Year	A	B	C
1	$2,000	$2,000	$ 5,000
2	3,000	2,000	5,000
3	4,000	2,000	(5,000)
4	(5,000)	2,000	(5,000)
5	5,000	5,000	15,000

What is the present value of each of these three investments if the appropriate discount rate is 10 percent?

6–51. **(Calculating the present value of complex cash flows)** You have an opportunity to make an investment that will pay $100 at the end of the first year, $400 at the end of the second year, $400 at the end of the third year, $400 at the end of the fourth year, and $300 at the end of the fifth year.

a. Find the present value if the interest rate is 8 percent. (Hint: You can simply bring each cash flow back to the present and then add them up. Another way to work this problem is to use either the = NPV function in Excel or the CF key on your financial calculator—but you'll want to check your calculator's manual before you use this key. Keep in mind that with the = NPV function in Excel, there is no initial outlay. That is, all this function does is bring all of the future cash flows back to the present. With a financial calculator, you should keep in mind that CF_0 is the initial outlay or cash flow at time 0 and, because there is no cash flow at time 0, $CF_0 = 0$.)

b. What would happen to the present value of this stream of cash flows if the interest rate was 0 percent?

6–52. **(Calculating the present value of complex cash flows)** How much do you have to deposit in an account paying 8 percent per annum if you want to withdraw £15,000 at the end of Year 5 and then £5,000 each year for next five years (from Year 6 to Year 10)?

6–53. **(Solving a comprehensive problem)** You would like to have $50,000 in 15 years. To accumulate this amount, you plan to deposit an equal sum in the bank each year that will earn 7 percent interest compounded annually. Your first payment will be made at the end of the year.

a. How much must you deposit annually to accumulate this amount?

b. If you decide to make a large lump-sum deposit today instead of the annual deposits, how large should this lump-sum deposit be? (Assume you can earn 7 percent on this deposit.)

c. At the end of 5 years, you will receive $10,000 and deposit this in the bank toward your goal of $50,000 at the end of 15 years. In addition to this deposit, how much must you deposit in equal annual deposits to reach your goal? (Again, assume you can earn 7 percent on this deposit.)

6–54. **(Calculating complex annuity payments)** Milhouse, 22, is about to begin his career as a rocket scientist for a NASA contractor. Being a rocket scientist, Milhouse knows that he should begin saving for retirement immediately. Part of his inspiration came from reading an article on Social Security in *Time*. The article indicated that the ratio of workers paying taxes to retirees collecting checks will drop dramatically in the future. In fact, the number will drop to two workers for every retiree in 2040. Milhouse's retirement plan allows him to make equal yearly contributions, and it pays 9 percent interest annually. Upon retirement, Milhouse plans to buy a new boat, which he estimates will cost him $300,000 in 43 years, which is when he plans to retire (at age 65). He also estimates that in order to live comfortably he will require a yearly income of $80,000 for each year after he retires. Based on his family history, Milhouse expects to live until age 80 (that is, he would like to receive a payment of $80,000 at the end of each year for 15 years). When he retires, Milhouse will purchase his boat in one lump sum and place the remaining balance into an account that pays 6 percent interest, from which he will withdraw his $80,000 per year. If Milhouse's first contribution is made one year from today and his last is made the day he retires, how much money must he contribute each year to his retirement fund?

6–55. **(Solving a comprehensive problem)** Having just inherited a large sum of money, you are trying to determine how much you should save for retirement and how much you can spend now. For retirement, you will deposit today (January 1, 2016) a lump sum in a bank account paying 10 percent compounded annually. You don't plan on touching this deposit until you retire in five years (January 1, 2021), and you plan on living for 20 additional years. During your retirement, you would like to receive a payment of $50,000 on the first day of each year, with the first payment on January 1, 2021, and the last payment on January 1, 2041. Complicating this objective is your desire to have one final three-year fling during which time you'd like to track down all the original cast members of *Hey Dude* and *Saved by the Bell* and get their autographs. To finance this, you want to receive $250,000 on January 1, 2036, and *nothing* on January 1, 2037, and January 1, 2038, because you will be on the road. In addition, after you pass on (January 1, 2041), you would like to have a total of $100,000 to leave to your children.

a. How much must you deposit in the bank at 10 percent interest on January 1, 2016, to achieve your goal? (Use a timeline to answer this question. Keep in mind that the last second of December 31 is equivalent to the first second of January 1.)

b. What kinds of problems are associated with this analysis and its assumptions?

6–56. **(Calculating the future value of a complex annuity)** Springfield mogul Montgomery Burns, age 80, wants to retire at age 100 so he can steal candy from babies full-time. Once Mr. Burns retires, he wants to withdraw $1 billion at the beginning of each year for 10 years from a special offshore account that will pay 20 percent annually. In order to fund his retirement, Mr. Burns will make 20 equal end-of-the-year deposits in this same special account that will pay 20 percent annually. How much money will Mr. Burns need at age 100, and how large of an annual deposit must he make to fund this retirement amount?

6–57. **(Solving a comprehensive problem)** Suppose that you are in the fall of your senior year and are faced with the choice of either getting a job when you graduate or going to law school. Of course, your choice is not purely financial. However, to make an informed decision you would like to know the financial implications of the two alternatives. Let's assume that your opportunities are as follows:

- If you take the "get a job" route, you expect to start off with a salary of $40,000 per year. There is no way to predict what will happen in the future, but your best guess is that your salary will grow at 5 percent per year until you retire in 40 years.
- As a law student, you will be paying $25,000 per year in tuition for each of the three years you are in graduate school. However, you can then expect a job with a starting salary of $70,000 per year. Moreover, you expect your salary to grow by 7 percent per year until you retire 35 years later.

Clearly, your total expected lifetime salary will be higher if you become a lawyer. However, the additional future salary is not free. You will be paying $25,000 in tuition at the beginning of each of the three years of law school. In addition, you will be giving up a little more than $126,000 in lost income over the three years of law school: $40,000 the first year, $42,000 the second year, and $44,100 the third year.

a. To start your analysis of whether to go to law school, calculate the present value of the future earnings that you will realize by going directly to work, assuming a 3 percent discount rate.

b. What is the present value of your future earnings if you decide to attend law school, assuming a 3 percent discount rate? Remember that you will be in law school for three years before you start to work as a lawyer. (Hint: Assume you are paid at the end of each year, so that if you decide to go to law school, your first salary payment occurs four years from now.)

c. If you pay your law school tuition at the beginning of each year, what is the present value of your tuition, assuming a 3 percent discount rate?

6–58. **(Calculating the present value of a complex stream)** Don Draper has signed a contract that will pay him $80,000 at the *end* of each year for the next six years, plus an additional $100,000 at the end of Year 6. If 8 percent is the appropriate discount rate, what is the present value of this contract?

6–59. **(Calculating the present value of a complex stream)** Don Draper has signed a contract that will pay him $80,000 at the *beginning* of each year for the next six years, plus an additional $100,000 at the end of Year 6. If 8 percent is the appropriate discount rate, what is the present value of this contract?

6–60. **(Analyzing a complex stream of cash flows)** Roger Sterling has decided to buy an ad agency and is going to finance the purchase with seller financing—that is, a loan from the current owners of the agency. The loan will be for $2,000,000 financed at a 7 percent nominal annual interest rate. Sterling will pay off the loan over five years with end-of-month payments along with a $500,000 lump-sum payment at the end of Year 5. That is, the $2 million loan will be paid off with monthly payments, and there will also be a final payment of $500,000 at the end of the final month. How much will the monthly payments be?

Mini-Case

Barry Neill, 60, has just retired from his job at the city council, where he has worked for 38 years. He has a family that is dependent on him. He has received a lump sum retirement bonus of £80,000 in addition to £1,700 per month as his pension. Besides this lump sum bonus, he also has £60,000 in an ISA account and owns mutual fund units worth £80,000 as of now. The ISA account pays 4 percent per annum.

Barry has estimated that he will need around £3,700 per month to maintain his family and lifestyle. He will be eligible for state pension of £300 a month once he turns 65. However, if he defers withdrawing his state pension until he is 70, he will get £560 a month.

Advise Barry on what he should do with his receipts from pension funds and savings.

Questions

1. Barry has access to some money for emergencies from his wife's family. He can thus use his overall receipts from his savings—£220,000—for retirement. If his savings pay him 5 percent per annum and he expects to live for another 35 years, how much can he withdraw on a monthly basis?
2. Ignoring his state pension, is the amount determined in Question 1 sufficient for meeting Barry's monthly expenses in addition to his pension of £1,700? If not, how long will his retirement savings last if his current expenditure remains the same? What if he reduces his expenditure to £3,200?
3. Considering the information obtained in Question 2, should Barry wait till he is 65 to obtain his state pension? If he waits until the age of 70, how will this state pension change the answer to Question 2?
4. If the inflation rate averages 3 percent during his retirement, how old will he be when the prices have doubled from current levels? How much will a newspaper cost when he is 95 years old, in 35 years' time, if it costs £1.25 today?