Chapter 3 Renewal Theory

3.1. Introduction

A Poisson process can be defined as a counting process for which the interarrival times are iid with an exponential distribution. A renewal process is more general counting process than Poisson process, which is defined as below.

Definition 3.1. (Renewal Process) A counting process for which the interarrival times are iid with an arbitrary distribution is said to be a renewal process.

An event is called a renewal if upon its occurrence everything starts over again probabilistically. Let X_1 be the time to the first renewal and let X_n (n=2,3,…) be the time between (n-1)st renewal and n-th renewal. Assume that X_n (n=1,2,…) are iid random variables with distribution function of F. To be nontrivial, assume that

$$F(0) = P\{X_n = 0\} < 1.$$

Let

$$\mu = E[X_n] = \int_0^\infty x dF(x)$$

which will be positive. Define the time of the n-th renewal by

$$S_n = \sum_{i=1}^{n} X_i$$

Let N(t) be the number of renewals by time t so that

$$N(t) = \max\{n : S_n \le t\}.$$

Then, the counting process {N(t), t≥0} will be a renewal process.

Example 3.1. Consider a component that is used continuously with replacements. Let Y be the lifetime of the component, which is random with distribution function of G. The component is replaced by a new one upon failure or at a fixed time period T, whichever comes first. (This replacement policy is called an age replacement.) Then, each replacement will be a renewal and so counting the number of replacements leads to a renewal process. An interarrival time X will be Y or T depending on whether the lifetime is shorter than T or not. That is,

$$X = \min(Y,T) = \begin{cases} Y & if\ Y < T \\ T & if\ Y \ge T \end{cases}$$

The mean interarrival time is obtained by

$$\mu = E[\min(Y,T)] = \int_0^\infty P\{\min(Y,T) > x)dx = \int_0^T P\{Y > x\}dx = \int_0^T G^c(x)dx.$$

3.2. Distribution of the Number of Renewals

Suppose we are interested in the distribution of N(t). The following result holds.

Proposition 3.1. For a renewal process {N(t), t≥0} with interarrival time distribution of F,

$$P\{N(t)=n\} = F^{(n)}(t) - F^{(n+1)}(t),\ n = 0,1,...$$

where $F^{(n)}$ is n-fold convolution of F with $F^{(0)} = 1$.

(proof) The event {N(t) ≥ n} is equivalent to the event $\{S_n \le t\}$. So,

$$P\{N(t) \ge n\} = P\{S_n \le t\} = F^{(n)}(t).$$

Therefore,

$$P\{N(t)=n\} = P\{N(t)\ge n\} - P\{N(t) \ge n+1\} = F^{(n)}(t) - F^{(n+1)}(t).$$

Definition 3.2. Let m(t) = E[N(t)]. Then m(t) is called a *renewal function*.

Proposition 3.2. The renewal function is given by

$$m(t) = \sum_{n=1}^{\infty} F^{(n)}(t).$$

(proof)

$$m(t) = E[N(t)] = \sum_{n=1}^{\infty} P\{N(t) \ge n\} = \sum_{n=1}^{\infty} F^{(n)}(t).$$

The second equality holds since N(t) is a nonnegative random variable.

Example 3.2. Consider a renewal process whose interarrival time follows iid with uniform distribution between 0 and 1. Find the renewal function for 0 ≤ t ≤ 1.

(Solution) For 0 ≤ t ≤ 1 the n-fold convolution of F is given by

$$F^{(n)}(t) = \frac{t^n}{n!},\ \text{n=1,2,…}$$

which can be shown from the mathematical induction. Note that for 0 ≤ t ≤ 1

$$\begin{aligned} F^{(n+1)}(t) = P\{S_{n+1} \le t\} &= \int_0^t P\{S_{n+1} \le t \mid S_n = x\} \frac{x^{n-1}}{(n-1)!} dx \\ &= \int_0^t P\{X_{n+1} \le t - x\} \frac{x^{n-1}}{(n-1)!} dx \\ &= \int_0^t (t-x) \frac{x^{n-1}}{(n-1)!} dx \\ &= \frac{t^{n+1}}{(n+1)!} \end{aligned}$$

Therefore, the renewal function for 0 ≤ t ≤ 1 is given by

$$m(t)=\sum_{n=1}^{\infty}F^{(n)}(t)=\sum_{n=1}^{\infty}\frac{t^n}{n!}=e^t-1.$$

So, the mean number of renewals to occur by time t=0.5 is 0.6487 and the mean number by t=1 is 1.7183 while the mean interarrival time is 0.5.

It can be shown that the renewal function m(t), $0 \le t < \infty$, uniquely determines the interarrival time distribution F. For example, m(t)=λt corresponds to the exponential distribution with mean 1/λ.

Suppose that we are interested in the time of the first renewal after t, which is

$$S_{N(t)+1}=\sum_{i=1}^{N(t)+1}X_i$$

The expected value of this may not be easily obtained since N(t) is dependent on the sequence of X_i's. We already know that if X_i are iid and N is independent of X_i's then

$$E\left[\sum_{i=1}^{N}X_i\right]=E[N]E[X]$$

Even if N is not independent of X_i's but N is a stopping time of $\{X_i, i=1,2,\ldots\}$ then the Wald's equation states that the above equality holds.

Definition 3.3. (Stopping time) An integer-valued random variable N is said to be a stopping time for the sequence of independent random variables, $X_1, X_2, \ldots$ if the event {N=n} is independent of $X_{n+1}, X_{n+2}, \ldots$ for all n=1,2,….

Example 3.3 (a). Let X_n, n=1, 2,… be independent and $P\{X_n=0\}=P\{X_n=1\}=1/2$. Also, let

$$N=\min\{n: X_1+\ldots+X_n=10\}$$

Then, obviously N is a stopping time of $\{X_i, i=1,2,\ldots\}$.

(b) Let X_n, n=1, 2,… be independent and let

$$N=\max\{n: X_n\ge 5\}$$

Then, N is not a stopping time.

Example 3.4 (a) For a renewal process {N(t), t≥0}, N(t)+1 is a stopping time for interarrival times X_i's. It can be seen that the following events are equivalent:

$$\{N(t)+1=n\}\equiv\{N(t)=n-1\}\equiv\{X_1+\ldots+X_{n-1}\le t \text{ and } X_1+\ldots+X_n>t\}$$

So, {N(t)+1=n} depends only on $X_1,\ldots,X_n$ and is independent of $X_{n+1}, X_{n+2},\ldots$

(b) For a renewal process {N(t), t≥0}, N(t) is *not* a stopping time for interarrival times X_i's. The reasoning may be similar to (a).

Theorem 3.3. (Wald's Equation) If $X_1, X_2, \ldots$ are iid random variables with E[X] < ∞ and if N is a stopping time for $X_1, X_2, \ldots$ such that E[N] < ∞, then

$$E\left[\sum_{i=1}^{N} X_i\right] = E[N]E[X].$$

(proof) Let for n=1,2,…

$$I_n = \begin{cases} 1 & \text{if } n \le N \\ 0 & \text{if } n > N \end{cases}$$

then,

$$\sum_{n=1}^{N} X_n = \sum_{n=1}^{\infty} X_n I_n$$

Hence,

$$E\left[\sum_{n=1}^{N} X_n\right] = E\left[\sum_{n=1}^{\infty} X_n I_n\right] = \sum_{n=1}^{\infty} E[X_n I_n]$$

Since N is a stopping time the event {N=n} or {I_n = 1} depends only on $X_1, \ldots, X_{n-1}$ and is independent of X_n. So, I_n is independent of X_n. Therefore,

$$E\left[\sum_{n=1}^{N} X_n\right] = \sum_{n=1}^{\infty} E[X_n]E[I_n] = E[X]\sum_{n=1}^{\infty} P\{N \ge n\} = E[X]E[N].$$

Proposition 3.4. The expected time of the first renewal after t is given by

$$E[S_{N(t)+1}] = \mu(m(t)+1)$$

(proof) The result immediately follows from the Wald's equation since N(t)+1 is a stopping time for X_i's.

Note that we cannot apply Wald's equation to obtain $E[S_{N(t)}]$ since N(t) is not a stopping time. In fact, the interarrival time containing time t, $X_{N(t)+1}$, has a different distribution from the usual ones. We will consider this quantity (called spread at t) later.

Example 3.5. (Poisson process) Suppose that {N(t), t≥0} is a Poisson process having rate λ. Then,

$$E[S_{N(t)+1}] = \mu(m(t)+1) = \frac{1}{\lambda}(\lambda t + 1) = t + \frac{1}{\lambda}$$

which is also intuitively derived since $S_{N(t)+1}$ is t plus time to next event. But,

$$E[S_{N(t)}] = E[E[S_{N(t)} \mid N(t)]]$$

$$= E\left[t\frac{N(t)}{N(t)+1}\right]$$

The second equality holds since $S_{N(t)} \mid N(t) = n$ is the largest one among a sample of size n from Unif(0, t). Using the property that

$$E\left[\frac{Y}{Y+1}\right]=E\left[\frac{Y-1}{\alpha}\right] \text{ for } Y \sim Poi(\alpha)$$

we have

$$E[S_{N(t)}]=E\left[t\frac{N(t)-1}{\lambda t}\right]=t-\frac{1}{\lambda}$$

which is different from $\mu m(t)=t$.

Def 3.4. Suppose that {N(t), t≥0} is a renewal process.

(a) The age at t of the renewal process is defined by

$$A(t)=t-S_{N(t)}$$

(b) The excess at t of the renewal process is defined by

$$Y(t)=S_{N(t)+1}-t$$

(c) The spread at t of the renewal process is defined by

$$X_{N(t)+1}=A(t)+Y(t)$$

Our major interests regarding a renewal process is to obtain long-run properties. In fact, two types of long-run properties will be analyzed. One is to obtain the long-run (time) average of the quantity of interest and the other is to obtain the pointwise limit. For example, the long-run average of age in Def. 3.4 will be

$$\lim_{t\to\infty}\frac{\int_0^t A(s)ds}{t}$$

while the pointwise limit of the expected age will be

$$\lim_{t\to\infty}E[A(t)].$$

The Section 3.3 and Section 3.4 deal with the long-run average and the Section 3.5 and Section 3.6 will describe how to obtain the pointwise limit of the quantity of interest.

3.3. Long-run Renewal Rate

This section deals with the average number of renewals (per unit time) in the long run, which will be called a long-run renewal rate.

Prop 3.5. For a renewal process {N(t), t≥0} having df F for interarrival times,

$$\lim_{t\to\infty}\frac{N(t)}{t}=\frac{1}{\mu} \quad \text{w.p. 1}$$

where

$$\mu=\int_0^\infty xdF(x)$$

(proof) Since $S_{N(t)}$ is the last renewal time prior to t and $S_{N(t)+1}$ is the first renewal time after t,

$$S_{N(t)} \le t < S_{N(t)+1} \quad \text{or}$$

$$\frac{S_{N(t)}}{N(t)} \le \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

But,

$$\lim_{t\to\infty} \frac{S_{N(t)}}{N(t)} = \lim_{t\to\infty} \frac{X_1 + ... + X_{N(t)}}{N(t)} = E[X] = \mu \quad \text{w.p. 1}$$

and

$$\lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)} = \lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)+1} \frac{N(t)+1}{N(t)} = \mu \quad \text{w.p. 1}$$

So the result follows.

Theorem 3.6. (Elementary Renewal Theorem) For a renewal process {N(t), t≥0} having μ of mean interarrival time

$$\lim_{t\to\infty} \frac{E[N(t)]}{t} = \lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$$

Note that Theorem 3.6 can not be just derived from Prop. 3.5. In general, the limit value of a sequence of random variables is not same as the limit value of a sequence of means of random variables.

Example 3.6. Let a random variable U follow Unif(0, 1) and let

$$Y_n = \begin{cases} 0 & \text{if } U > 1/n \\ n & \text{if } U \le 1/n \end{cases}, \quad n = 1,2,...$$

Then,

$$\lim_{n\to\infty} Y_n = 0 \quad \text{w.p.1}$$

But,

$$E[Y_n] = nP\{U \le 1/n\} = n(1/n) = 1$$

Example 3.7. (M/G/1 Loss) Customers arrive at a telephone booth according to a Poisson process having rate λ. However, a customer only enters the booth when it is empty. So, no queues are allowed. The service time (time duration that a customer occupies the booth) follows df G. This type of queueing system is called *M/G/1 loss* system.

(a) What is the rate at which customers enter the booth ?

(b) What proportion of potential customers cannot enter the booth (called loss rate) ?

(solution) A renewal occurs every time that a customer actually enters the booth. If N(t) denotes the number of customers who enter the booth by t, then {N(t), t≥0} will be a renewal process having mean interarrival time of

$$\mu = \text{mean service time} + \text{mean time to next customer's arrival}$$

Here, the time to the next customer follows exponential with rate λ from the memoryless property. So,

$$\mu = \mu_G + \frac{1}{\lambda}$$

where

$$\mu_G = \int_0^\infty x dG(x) .$$

(a) Since the rate at which customers enter the booth is just the long-run renewal rate, it is given by

$$\text{rate at which customers enter} = \frac{1}{\mu} = \frac{\lambda}{1+\lambda\mu_G}$$

(b) The loss rate of customers is obtained by

$$\text{loss rate} = 1 - \text{proportion of customers that enter}$$

Here,

$$\text{proportion of customers that enter} = \frac{\text{rate at which customers enter}}{\text{arrival rate}}$$
$$= \frac{\lambda/(1+\lambda\mu_G)}{\lambda} = \frac{1}{1+\lambda\mu_G}$$

So,

$$\text{loss rate} = \frac{\lambda\mu_G}{1+\lambda\mu_G}$$

3.4. Renewal Reward Processes

Consider a renewal process {N(t), t≥0}. Let us assume that reward will be earned at the time of renewal. The reward can be cost or profit attached to the renewal. Let R_n denote the reward earned at the time of n-th renewal (n=1,2,…), which are iid random variables having a common mean E[R]. R_n may depend on X_n. Then, the total reward earned by t, R(t), will be

$$R(t) = \sum_{n=1}^{N(t)} R_n .$$

The new process {R(t), t≥0} is called a *renewal reward process*.

Prop. 3.7. Suppose that {R(t), t≥0} is called a renewal reward process having E[R] < ∞ and E[X] < ∞. Then, the long-run reward rate or long-run average reward is given by

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E[R]}{E[X]} \quad \text{w.p.1.}$$

(proof) Since

$$\frac{R(t)}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \frac{N(t)}{t}$$

the result follows if SLLN is applied to each term.

The interarrival times in a renewal process is often called as renewal cycles. The above result says that the long-run reward rate is obtained by

$$\text{long - run reward rate} = \frac{E[\text{reward during a cycle}]}{E[\text{renewal cycle length}]}.$$

Prop. 3.8. Suppose that {R(t), t≥0} is called a renewal reward process having E[R] < ∞ and E[X] < ∞. Then,

$$\lim_{t\to\infty}\frac{E[R(t)]}{t} = \frac{E[R]}{E[X]}.$$

Example 3.8. (Age Replacement Model) Consider a component that is used continuously with replacements. Let Y be the lifetime of the component, which is random with distribution function of G. The component is replaced by a new one upon failure or at a fixed time period T, whichever comes first. This replacement policy is called an age replacement. The cost of a new component is c_1 and the additional cost incurred by a failure is c_2. Obtain the long-run average cost.

(solution) If we let N(t) be the number of replacements of components by t and let R(t) be the amount of cost incurred by t, then, {R(t), t≥0} becomes a renewal-reward process. The expected cycle length is given from Example 3.1. by

$$E[\text{cycle length}] = E[\min(Y,T)] = \int_0^T G^c(x)dx$$

The reward or cost during a cycle, R, is expressed by

$$R = \begin{cases} c_1 & \text{if } Y > T \\ c_1 + c_2 & \text{if } Y \le T \end{cases}$$

So, its expected value is obtained by

$$E[R] = c_1 P\{Y > T\} + (c_1 + c_2)P\{Y \le T\} = c_1 + c_2 G(T)$$

Hence, the long-run average cost is given by

$$\text{long - run average cost} = \frac{E[\text{cost during a cycle}]}{E[\text{cycle length}]} = \frac{c_1 + c_2 G(T)}{\int_0^T G^c(x)dx}$$

Note that the long-run average cost a function of T and so the optimal value which minimized the long-run average cost can be determined by differentiation.

Example 3.9. Jobs arrive at a service person according to a Poisson process having rate λ. Assume that each job has random value whose distribution function is F. The service person only accept a job when he/she is idle and the job has value greater than *v*. The service time for an accepted job follows df G. Obtain the long-run average reward.

(solution) The renewal cycle is said to begin whenever a job (no matter what value it has) arrives when the server is idle. Then, the renewal cycle is expressed by

$$\text{renewal cycle} = \begin{cases} \text{time to next job} & \text{if value} \le v \\ \text{time to next job} + \text{service time} & \text{if value} > v \end{cases}$$

So,

$$E[\text{renewal cycle}] = \frac{1}{\lambda} + \mu_G F^c(v)$$

where

$$\mu_G = \int_0^\infty x dG(x) .$$

The expected reward during a cycle is obtained by

$$E[\text{reward}] = E[\text{value} \mid \text{value} > v] P\{\text{value} > v\} = \int_v^\infty x dF(x) .$$

Hence, the long-run average reward is given by

$$\text{long-run average reward} = \frac{E[\text{reward}]}{E[\text{renewal cycle}]} = \frac{\int_v^\infty x dF(x)}{1/\lambda + \mu_G F^c(v)}$$

Example 3.10. (Average Age) The long-run average age or just average age is given by

$$\text{average age} = \lim_{t \to \infty} \frac{\int_0^t A(s) ds}{t}$$

Assuming that we are earning rewards at a rate equal to the age, the numerator represents the total reward earned by t. So, if we apply Prop. 3.7. then,

$$\text{average age} = \frac{E[\text{reward during a cycle}]}{E[X]} = \frac{E\left[\int_0^X s ds\right]}{E[X]} = \frac{E[X^2]}{2E[X]}$$

Prop. 3.9. The average spread is greater than or equal to the mean interarrival time of a renewal process. (This is called inspection paradox.)

(proof) It can be easily shown that the average excess is same as the average age given in Example 3.10. So, we need to show that

$$\text{average spread} = \frac{E[X^2]}{E[X]} \ge E[X] = \text{mean interarrival time}$$

The above follows since the variance of X is nonnegative.

3.5. Renewal Equations

Renewal equations are useful for deriving the quantity of interest associated with a renewal process as a function of time. A renewal equation is expressed by a recursive form through an integral equation. The solution to a renewal equation can be easily obtained.

To introduce the form of a renewal equation, let us consider the renewal function:

$$m(t) = E[N(t)]$$

which is an already known quantity. This can be evaluated by conditioning on X_1, the time of the first renewal:

$$m(t) = \int_0^\infty E[N(t) \mid X_1 = x] dF(x) .$$

The integrand can be evaluated by dividing into two cases: one is the case where the first renewal occurs after time t and the other is the case where the first renewal occurs before time t. The former case follows

$$E[N(t) \mid X_1 = x > t] = 0$$

since there are no renewals observed by t. The latter case follows

$$E[N(t) \mid X_1 = x \in [0,t]] = 1 + m(t-x)$$

since the first renewal occurs by t and the expected number of renewals between time x and t will be m(t-x) from the definition of the renewal function. So,

$$m(t) = \int_0^t \{1 + m(t-x)\} dF(x)$$

$$= F(t) + \int_0^t m(t-x) dF(x)$$

The above form of equation is called a renewal equation if m(.) is considered as unknown.

Suppose that Z(t) is an unknown function associated with a renewal process with distribution function of F. Then the general form of renewal equation is given by

$$Z(t) = Q(t) + \int_0^t Z(t-x) dF(x)$$

where Q(t) is a known function. If we write the second term as

$$Z * F(t) = \int_0^t Z(t-x) dF(x)$$

then the renewal equation can be given by

$$Z(t) = Q(t) + Z * F(t)$$

Proposition. The solution to the renewal equation in Eq.( ) is given by

$$Z(t) = Q(t) + \int_0^t Q(t-x) dm(x)$$

where m(t) is the renewal function.

(Proof) We need to show that

$$Z * F(t) = Q * m(t)$$

If we convolve Eq.( ) with F, then we have

Z*F(t)=Q*F(t) + Q*m*F(t)

= Q*F(t) + Q*(m(t)-F(t))

= Q*F(t) + Q*m(t) – Q*F(t)

= Q*m(t).

Note that $dm(x)$ can be interpreted as the probability that a renewal occurs in $(x, x+dx)$ since

$$dm(x) = \sum_{n=1}^{\infty} dF^{(n)}(x) = \sum_{n=1}^{\infty} f^{(n)}(x)dx = \sum_{n=1}^{\infty} P\{n^{th} \text{ renewal occurs in } (x,\ x+dx)\}\ .$$

In summary, a quantity of interest associated with a renewal process, Z(t) say, can be obtained through the following *four-step approach*:

1) Condition Z(t) on X_1.
2) Evaluate by renewal arguments the integrand or conditional Z(t) on two cases of X_1: for X_1>t and for X_1≤t.
3) Uncondition to derive the renewal equation for Z(t).
4) Obtain the solution to the renewal equation to find Z(t).

Example (Mean Excess at t) Let Y(t) be the excess at time t. Let us find the following mean excess:

g(t) = E[Y(t)].

1) Conditioning on X_1 gives

$$g(t) = \int_0^{\infty} E[Y(t) \mid X_1 = x]dF(x).$$

2) The integrand for X_1>t will be

$$E[Y(t) \mid X_1 = x > t] = x - t$$

while for X_1≤t it will be

$$E[Y(t) \mid X_1 = x \in [0,t]] = g(t-x)$$

since upon the first renewal everything starts over again.

3) So,

$$g(t) = \int_t^{\infty} (x-t)dF(x) + \int_0^t g(t-x)dF(x)$$

If we let

$$h(t) = \int_t^{\infty} (x-t)dF(x)$$

then we have the following renewal equation for g(t):

$$g(t) = h(t) + \int_0^t g(t-x)dF(x)$$

4) The solution for g(t) is given by

$$g(t) = h(t) + \int_0^t h(t-x)dm(x) .$$

Example (Distribution of Excess at t) Obtain the df of excess at t.

(solution) Let

$$H_y(t) = P\{Y(t) \le y\} .$$

The four-step approach is as follows:

1) $H_y(t) = \int_0^\infty P\{Y(t) \le y \mid X_1 = x]dF(x).$

2) For X_1 >t

$$P\{Y(t) \le y \mid X_1 > t+y\} = 0$$
$$P\{Y(t) \le y \mid X_1 \in (t, t+y)\} = 1$$

For X_1 ≤t

$$P\{Y(t) \le y \mid X_1 = x \in (0,t)\} = H_y(t-x)$$

3) $H_y(t) = \int_t^{t+y} dF(x) + \int_0^t H_y(t-x)dF(x) = F(t+y) - F(t) + \int_0^t H_y(t-x)dF(x)$

4) $H_y(t) = F(t+y) - F(t) + \int_0^t [F(t+y-x) - F(t-x)\}dm(x)$

3.6. Key Renewal Theorem

In order to obtain the pointwise limit of a quantity of interest we first derive the quantity as a function of time and take a limit. Suppose that we are interested in the pointwise limit of Z(t) in Eq.( ). Then,

$$\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} Q(t) + \lim_{t\to\infty} \int_0^t Q(t-x)dm(x) .$$

The Key Renewal Theorem is to obtain the second term in the right-hand side. The existence of the limit requires some conditions. The following definitions are needed to describe these conditions.

Definition . A nonnegative random variable X is said to be *lattice* if there exists d ≥ 0 such that

$$\sum_{n=0}^{\infty} P\{X = nd\} = 1 .$$

The largest d having this property is said to be the *period* of X.

If X is lattice and X has a distribution function of F, then we say that F is lattice.

Example

(a) If X follows Poisson distribution with mean λ, then X is lattice with period 1.

(b) If P{X=4}=P{X=8}=P{X=12}=P{X=14}=1/4, then X is lattice with period 2.

(c) If P{X=2π}=1/3 and P{X=6π}=2/3, then X is lattice with period 2π.

(d) If P{X= $\sqrt{2}$ }=P{X= $\sqrt{3}$ }=1/2, then X is not lattice.

Theorem (Key Renewal Theorem) If F is not lattice and Q(t) is directly Riemann integrable, then

$$\lim_{t\to\infty}\int_0^t Q(t-x)dm(x) = \frac{1}{\mu}\int_0^\infty Q(t)dt$$

where m(t) is the renewal function and

$$\mu = \int_0^\infty x dF(x).$$

Sufficient conditions for Q(t) to be directly Riemann integrable are:

1) Q(t) ≥ 0
2) Q(t) is nonincreasing
3) $\int_0^\infty Q(t)dt < \infty$

Example (Limiting Mean Excess) Let Y(t) be the excess at t. The mean excess is obtained in Example by

$$E[Y(t)] = h(t) + \int_0^t h(t-x)dm(x)$$

where

$$h(t) = \int_t^\infty (x-t)dF(x).$$

So,

$$\lim_{t\to\infty} E[Y(t)] = \lim_{t\to\infty} h(t) + \lim_{t\to\infty}\int_0^t h(t-x)dm(x)$$
$$= \lim_{t\to\infty}\int_0^t h(t-x)dm(x)$$

If we assume that the second moment of an interarrival time (X, say) is finite, then the function h(t) is directly Riemann integrable. If we assume that F is not lattice in addition, we can apply Key Renewal Theorem to obtain the limiting mean excess as follows:

$$\lim_{t\to\infty} E[Y(t)] = \lim_{t\to\infty}\int_0^t h(t-x)dm(x)$$
$$= \frac{1}{\mu}\int_0^\infty h(t)dt$$
$$= \frac{1}{\mu}\int_0^\infty\int_t^\infty (x-t)dF(x)dt$$

If we change the order of two integrals, we have

$$\lim_{t\to\infty} E[Y(t)] = \frac{1}{\mu}\int_0^\infty \int_0^x (x-t)dt\, dF(x)$$
$$= \frac{1}{2\mu}\int_0^\infty x^2 dx$$
$$= \frac{E[X^2]}{2E[X]}$$

Note that the result is same as the average excess.

Theorem (Blackwell's Theorem)

1) If F is not lattice, then

$$\lim_{t\to\infty}[m(t+a)-m(t)] = \frac{a}{\mu}$$

2) If F is lattice with period d, then

$$\lim_{n\to\infty} E[\text{number of renewals at } nd] = \frac{d}{\mu}$$

Note that Blackwell's Theorem for lattice case states

$$\lim_{n\to\infty} P[\text{renewal occurs at } nd] = \frac{d}{\mu}$$

since the number of renewals at time *nd* will be 1 or 0.

Example. Suppose that interarrival times have the following distribution.

$$X = \begin{cases} 2 & w.p. 1/2 \\ 4 & w.p. 1/2 \end{cases}.$$

Then, X is lattice with d=2. So,

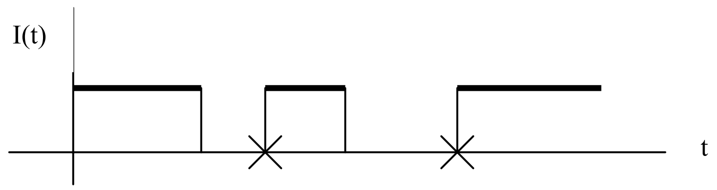
$$\lim_{n\to\infty} E[\text{number of renewals at } 2n] = \lim_{n\to\infty} P[\text{renewal occurs at } 2n] = \frac{2}{3}$$

3.7. Alternating Renewal Processes

Consider a renewal process whose interarrival times are N_s having the distribution F. Suppose that an interarrival time consists of an ON period and an OFF period such that $X_n = Z_n + Y_n$ where Z_n is the n-th ON period and Y_n is the n-th OFF period. Suppose also that Z_n are iid as H and Y_n are iid as G. Z_n and Y_n may not be independent.

Let

$$I(t) = \begin{cases} 1 \text{ } if \text{ } the \text{ } system \text{ } is \text{ } ON \text{ } at \text{ } t \\ 0 \text{ } otherwise \end{cases}$$



We are interested in

1) What is the long-run proportion of time that the system is ON ?

$$\lim_{t \to \infty} \frac{\int_0^t I(x)dx}{t} = ?$$

2) What is the limiting probability that the system is ON ?

$$\lim_{t\to\infty} P\{I(t)=1\} = ?$$

Theorem . The long-run proportion of time that the system is On is given by

$$\lim_{t \to \infty} \frac{\int_0^t I(x)dx}{t} = \frac{E[Z_n]}{E[X_n]} \quad \text{w.p. 1.}$$

(proof) Obvious from the result from the renewal-reward process.

Theorem . If $E[Z_n + Y_n] < \infty$ and F is non-lattice, then

$$\lim_{t\to\infty} P\{I(t)=1\} = \frac{E[Z_n]}{E[Z_n]+E[Y_n]}.$$

(proof)

Let P(t)=P{I(t)=1}. Then apply four-step approach to obtain P(t).

Condition on X_1. Then,

$$P(t) = \int_0^\infty P\{I(t)=1 \mid X_1 = x\}dF(x).$$

The integrand is evaluated as follows:

i) for X_1>t

$$P\{I(t)=1 \mid X_1 > t\} = P\{Z_1 > t \mid X_1 > t\} = \frac{P\{Z_1 > t\}}{P\{X_1 > t\}} = \frac{H^c(t)}{F^c(t)}$$

ii)for X_1<t

$$P\{I(t)=1 \mid X_1 = x \in (0,t)\} = P(t-x)$$

Therefore we have the following renewal equation:

$$P(t) = \frac{H^c(t)}{F^c(t)} F^c(t) + \int_0^t P(t-x)dF(x)$$
$$= H^c(t) + \int_0^t P(t-x)dF(x)$$

The solution to the above equation is given by

$$P(t) = H^c(t) + \int_0^t H^c(t-x)dm(x)$$

Hence by KRT,

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \int_0^t H^c(t-x)dm(x)$$
$$= \frac{1}{E[X_n]} \int_0^\infty H^c(x)dx$$
$$= \frac{E[Z_n]}{E[X_n]}.$$

Remarks:

1) $\lim_{t\to\infty} P\{I(t)=1\} = \frac{E[ON\ period]}{E[cycle\,length]}$ = long-run proportion of time that is ON

2) $\lim_{t\to\infty} P\{I(t)=0\} = \frac{E[OFF\ period]}{E[cycle\,length]}$.

Example . (Limiting distributions of Age/Excess/Spread) Consider an ordinary renewal process having the non-lattice distribution function of F with a finite mean for the interarrival times. Let A(t), Y(t) and $X_{N(t)+1}$ be the age, excess, and spread at time t of the renewal process. Obtain the limiting distributions of them, respectively. That is,

1) $\lim_{t\to\infty} P\{A(t) \le x\}$
2) $\lim_{t\to\infty} P\{Y(t) \le x\}$
3) $\lim_{t\to\infty} P\{X_{N(t)+1} \le x\}$

(Solution)

1) (Limiting Age Distribution) Suppose for a fixed x that the system is ON as long as the age is less than or equal to x. Let

$$I(t) = \begin{cases} 1 & if\ A(t) \le x \\ 0 & otherwise \end{cases}$$

Then, from Theorem 2,

$$\lim_{t\to\infty} P\{A(t) \le x\} = \lim_{t\to\infty} P\{I(t)=1\} = \frac{E[\text{ON period}]}{E[\text{cycle length}]}.$$

In a renewal cycle of length X “ON period” is the minimum of x and X. Note that the whole cycle is ON if X <

x. So,

$$E[\text{ON period}] = E[\min(x, X)] = \int_0^x F^c(y)dy .$$

Hence,

$$\lim_{t\to\infty} P\{A(t) \le x\} = \frac{1}{E[X]}\int_0^x F^c(y)dy$$

which is called an equilibrium distribution function of F. Note that we can derive this result through our four-step approach and applying Key Renewal Theorem.

2) (Limiting Excess Distribution) Suppose for a fixed x that the system is ON as long as the excess is less than or equal to x. In a renewal cycle larger than x OFF period appears first before ON period begins. Let

$$I(t) = \begin{cases} 1 & \text{if } Y(t) \le x \\ 0 & \text{otherwise} \end{cases}$$

Then, from Theorem 2,

$$\lim_{t\to\infty} P\{Y(t) \le x\} = \lim_{t\to\infty} P\{I(t) = 1\} = \frac{E[\text{ON period}]}{E[\text{cycle length}]} = \frac{1}{E[X]}\int_0^x F^c(y)dy ,$$

which is same as the limiting age distribution.

3) (Limiting Spread Distribution) Suppose that the system is ON when the spread is less than or equal to x. Let

$$I(t) = \begin{cases} 1 & \text{if } X_{N(t)+1} \le x \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$\lim_{t\to\infty} P\{Y(t) \le x\} = \lim_{t\to\infty} P\{I(t) = 1\} = \frac{E[\text{ON period}]}{E[\text{cycle length}]}$$

Note in this case that the whole cycle will be ON or OFF depending on cycle length ≤ x or cycle length > x. So,

$$\begin{aligned} E[\text{ON period}] &= E[\text{ON period} \mid X \le x]P\{X \le x\} + E[\text{ON period} \mid X > x]P\{X > x\} \\ &= E[X \mid X \le x]P\{X \le x\} \\ &= \int_0^x ydF(y) \end{aligned}$$

Therefore,

$$\lim_{t\to\infty} P\{X_{N(t)+1} \le x\} = \frac{1}{E[X]}\int_0^x ydF(y) .$$

Example (Busy Period in M/G/1 Queue) Consider an M/G/1 queue where customers arrive according to a Poisson process with rate λ to a one-server facility and service time of a customer follows the distribution function of G. A *busy period* is an interval that begins when an arrival finds the system empty and ends when, for the first time after that, a departure leaves the system empty. *Idle periods* are the intervals between successive busy periods. We are interested in the expected length of a busy period.

Successive busy periods and idle periods comprise an alternating renewal process since a renewal occurs whenever a busy period begins. Let B denote a busy period and I denote an idle period. Also let S denote the service time of a customer. Assume that λ E[S] < 1 to have a stable system. Then,

$$\text{proportion of time that system is busy} = \frac{E[B]}{E[B]+E[I]}$$

But, the followings are known to hold:

proportion of time that system is busy = P{server is busy}

= E[number of customers in service]

= λ E[S].

The last equality holds from Little's Law. The second last equality holds since there is one server. Therefore,

$$\lambda E[S] = \frac{E[B]}{E[B]+E[I]}$$

where E[I] = 1/λ from the memoryless property of an exponential distribution. So, we have

$$E[B] = \frac{E[S]}{1-\lambda E[S]}.$$

Example. (Inventory Model) Customers arrive at a store to purchase a certain product according to renewal process with interarrival distribution of F, non-lattice. The demand of a customer is random and follows distribution G independently of arrival times. This store adopts (s, S) policy. Derive the limiting distribution of the inventory level.

(Solution)

Let X_n be the sequence of interarrival times of customers having distribution of F and Y_n be the sequence of customer demands having distribution of G.

Also, let J(t) be the inventory level at time t. Assume that J(0)=S. Then, we would like to know

$$\lim_{t\to\infty} P\{J(t) \ge x\} = ?, s \le x \le S$$

We can say that a renewal occurs whenever the inventory level goes up to S. This occurs at the first time that the cumulative demand exceeds S-s. We divide a renewal cycle into ON period and OFF period according to whether $J(t) \ge x$ or not. Let

$$I(t) = \begin{cases} 1 & if\ J(t) \ge x \\ 0 & otherwise \end{cases}$$

Then, from Theorem 2, we have

$$\lim_{t\to\infty} P\{J(t) \ge x\} = \lim_{t\to\infty} P\{I(t) = 1\} = \frac{E[ON\ period]}{E[cycle\ length]}$$

Let

$$N_x = \min\{n : Y_1 + ... + Y_n > S - x\},$$

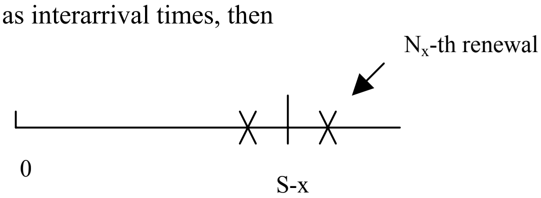
which represents the number of customers whose total demand first exceeds S-x. Note that a cycle length is the time until N_s customers arrive and that ON period is the time until N_x customers arrive.

Then,

$$\lim_{t\to\infty} P\{J(t)\ge x\} = \frac{E[ON\ period]}{E[cycle\ length]} = \frac{E\left[\sum_{i=1}^{N_x} X_i\right]}{E\left[\sum_{i=1}^{N_s} X_i\right]} = \frac{E[N_x]}{E[N_s]}$$

If we think of a renewal process having $Y_n, n = 1,2,...$ as interarrival times, then

$$E[N_x] = E[\#\,renewals\ by\ S-x]+1$$
$$= m_G(S-x)+1$$



where

$$m_G(t) = \sum_{n=1}^{\infty} G^{(n)}(t).$$

Therefore,

$$\lim_{t\to\infty} P\{J(t)\ge x\} = \frac{E[N_x]}{E[N_s]} = \frac{1+m_G(S-x)}{1+m_G(S-s)},\ s\le x\le S.$$

3.8. Delayed Renewal Processes

Consider a renewal process having distribution F for the interarrival times. Suppose we start observing the process from a certain time point.

Let

X_1 : time to the first renewal after observing

X_n (n=2,3,…): time between (n-1)st and n-th renewal

Then, X_1 has the different distribution G, say, from F for other interarrival times. Let us define the time of the n-th event as before (n=1,2,…):

$$S_n = \sum_{i=1}^{n} X_i$$

Let

$$N_D(t) = \max\{n : S_n \le t\}.$$

Then, $\{N_D(t), t\ge 0\}$ is said to be a *delayed (*or *general)* renewal process. Note that if G=F it will be an ordinary renewal process.

Example (Parallel System) A parallel system with three identical components is considered. The time to failure of a component is iid as exponential with rate λ and the time to repair is iid as exponential with rate μ. Note that the system breaks down whenever all three components are down.

Let N(t) be the number of times the system breaks down by time t. Then, we see that $\{N(t), t\ge 0\}$ is a delayed renewal process. Let X_1 be the time to the first system breakdown and X_2 be the time between the first and second breakdown. Then X_1 is the time until all three components are down plus the time to repair of any one component. However, X_n $(n \geq 2)$ is the time until all functioning (not necessarily all three) components are down plus the time to repair of any one component. Therefore, the distribution of X_1 is different from that of X_n $(n \geq 2)$.

The distribution of $N_D(t)$ is given by

$$P\{N_D(t) = n\} = P\{S_n \leq t\} - P\{S_{n+1} \leq t\} = G * F^{(n-1)}(t) - G * F^{(n)}(t), \quad n = 1,2,...$$

$$P\{N_D(t) = 0\} = P\{X_1 > t\} = G^C(t).$$

The renewal function is given by

$$m_D(t) = E[N_D(t)] = \sum_{n=1}^{\infty} G * F^{(n-1)}(t).$$

Theorem . The solution to the following renewal equation

$$H_D = Q_D + H * G$$

where H=Q+Q*m (solution to H=Q+H*F) is given by

$$H_D = Q_D + Q * m_D.$$

Proof

$$H_D = Q_D + H * G = Q_D + (Q + Q * m) * G = Q_D + Q * G + Q * m * G$$
$$= Q_D + Q * G + Q * (m_D - G) = Q_D + Q * m_D$$

Example (Distribution of Excess) Let $Y_D(t)$ be the excess at t of a delayed renewal process. Obtain the tail distribution of the excess.

(Solution)

Fix y and let Y(t) be the excess at t of an ordinary renewal process. Also let

$$H(t) = P\{Y(t) > y\}.$$

$$H_D(t) = P\{Y_D(t) > y\}.$$

Condition on X_1:

$$H_D(t) = \int_0^{\infty} P\{Y_D(t) > y \mid X_1 = x\} dG(x)$$

Evaluate the integrand:

$$P\{Y_D(t) > y \mid X_1 > t + y\} = 1$$

$$P\{Y_D(t) > y \mid X_1 = x \in (t, t + y)\} = 0$$

$$P\{Y_D(t) > y \mid X_1 = x \in (0, t)\} = H(t - x)$$

So, the renewal equation is

$$H_D(t) = G^C(t + y) + \int_0^t H(t - x) dG(x)$$

Solve the renewal equation using Theorem 1

$$H_D(t) = G^C(t + y) + \int_0^t F^C(t + y - x) dm_D(x)$$

Theorem . Suppose that $\{N_D(t), t \geq 0\}$ is a delayed renewal process having G for the time to the first renewal and F for the interarrival times thereafter. Let $\mu_F = \int x dF(x)$. Then,

1) $\lim_{t \to \infty} \frac{N_D(t)}{t} = \frac{1}{\mu_F} \; w.p.1.$

2) $\lim_{t \to \infty} \frac{m_D(t)}{t} = \frac{1}{\mu_F}.$

3) If F is not lattice, then for all $a \geq 0$

$$m_D(t+a) - m_D(t) \to \frac{a}{\mu_F} \quad as\ t \to \infty.$$

4) If F and G are lattice with period d, then

$$E[number\ of\ renewals\ at\ nd] \to \frac{d}{\mu_F} \quad as\ \ n \to \infty.$$

5) If F is not lattice, $\mu_F < \infty$ and h is directly Riemann integrable, then

$$\lim_{t \to \infty} \int_0^t h(t-x) dm_D(x) = \frac{1}{\mu_F} \int_0^\infty h(t) dt.$$

Example . A fair coin is tossed repeatedly and each outcome of H for head or T for tail is recorded sequentially. We are interested in the long-run rate at which the pattern "THTH" occurs.

Suppose for example that the sequence is

HTHTTHTHTHHHTHTHH…

↑ ↑ ↑

Then, the pattern is observed at n=8 for the first time, n=10 for the second time, n=16 for the third time and so on.

Let N(n) be the number of patterns observed by time n. Then $\{N(n), n \geq 1\}$ is a delayed renewal process since the distribution of the time to the first pattern is different from that of the other inter-pattern times.

The long-run rate at which the pattern occurs is obtained as follows:

From Theorem 1(1),

$$\text{Rate at which the pattern occurs} = \lim_{n \to \infty} \frac{N(n)}{n} = \frac{1}{E[time\ between\ patterns]}$$

Also from Theorem 1(4) with d=1

$$\frac{1}{E[time\ between\ patterns]} = \lim_{n \to \infty} E[number\ of\ renewals\ at\ n] = \lim_{n \to \infty} P\{pattern\ occurs\ at\ n\} = (1/2)^4.$$

Note that

E[time to the first pattern] = E[time to TH] + E[time from TH to THTH]

= E[time to TH] + E[time from THTH to THTH]

= $(1/2)^2 + (1/2)^4$.

Example (Limiting distribution of excess) Let $Y_D(t)$ be the excess at t of a delayed renewal process. Find the limiting tail distribution of the excess.

(Solution)

Since

$$P\{Y_D(t) > y\} = G^C(t+y) + \int_0^t F^C(t+y-x)dm_D(x)$$

from KRT

$$\lim_{t\to\infty} P\{Y_D > y\} = \frac{1}{\mu_F}\int_0^\infty F^c(t+y)dt = \frac{1}{\mu_F}\int_y^\infty F^c(x)dx = F_e^c(y)$$

So, the limiting distribution of the excess of a delayed renewal process is same as that of an ordinary renewal process.

Definition (Equilibrium Renewal Process). A delayed renewal process with $G = F_e$ is said to be a equilibrium renewal process, where F_e is the equilibrium distribution of F given by

$$F_e(x) = \frac{1}{\mu_F}\int_0^x F^c(y)dy.$$

Theorem. For the equilibrium renewal process the followings hold:

1) The equilibrium renewal function is given by

$$m_e(t) = \frac{t}{\mu_F}$$

2) The excess at t has the distribution of F_e for all t ≥0. That is,

$P\{Y_e(t) \le x\} = F_e(x)$ for all t ≥0.

3) It has stationary increments. That is, $N_e(s+t) - N_e(s)$ has the same distribution as $N_e(t)$.

Proof:

1) The Laplace transform of a general renewal function is given by

$$\tilde{m}_D(s) = \int_0^\infty e^{-st}dm_D(t) = \sum_{n=1}^\infty \int_0^\infty e^{-st}d(G * F^{(n-1)}(t)) = \sum_{n=0}^\infty \tilde{G}(s)[\tilde{F}(s)]^n = \frac{\tilde{G}(s)}{1-\tilde{F}(s)}.$$

Therefore, for $G = F_e$

$$\tilde{F}_e(s) = \frac{1-\tilde{F}(s)}{\mu_F s}$$

so it reduces to

$$\tilde{m}_e(s) = \frac{1}{\mu_F s}.$$

So, the result follows.

2) It immediately follows from Example 4.

3) We know that the time to the first renewal from time 0 follows F_e and that the time to the first renewal from s also follows F_e for all $s \geq 0$. So, the result follows.

Note: The equilibrium renewal process is also called the *stationary* renewal process or stationary point process.

3.9. Regenerative Processes

Definition. A stochastic process $\{X(t), t \geq 0\}$ with state space {0, 1, 2, …} is said to be a *regenerative process* if there exits a time S_1 such that the continuation of the process beyond S_1 is a probabilistic replica of the whole process starting at 0.

Note that the successive regenerative cycles constitute a renewal process.

Example

(a) An alternating renewal process having ON and OFF states is a regenerative process with state 1 and 0.

(b) If X(t) denotes the number of customers in system of M/G/1 queue, then $\{X(t), t \geq 0\}$ is a regenerative process with state {0, 1, 2, …}. The regenerative point is the time the arrival finds the system empty.

Theorem . Suppose that $\{X(t), t \geq 0\}$ is a regenerative process. If S_1 follows F, non-lattice and $E[S_1] < \infty$, then

$$P_j = \lim_{t\to\infty} P\{X(t) = j\} = \frac{E[\text{amount of time in state } j \text{ during a cycle}]}{E[\text{regenerative cycle}]}.$$

Proof.

Fix j and let P(t)=P{X(t)=j}.

Condition on S_1:

$$P(t) = P\{X(t) = j \mid S_1 > t\}P\{S_1 > t\} + \int_0^t P(t-x)dF(x)$$

$$= P\{X(t) = j, S_1 > t\} + \int_0^t P(t-x)dF(x)$$

Solving the above renewal equation gives

$$P(t) = P\{X(t) = j, S_1 > t\} + \int_0^t P\{X(t-x) = j, S_1 > t-x\}dm(x)$$

Applying KRT, we have

$$P_j = \frac{1}{E[S_1]}\int_0^\infty P\{X(t) = j, S_1 > t\}dt$$

To interpret the result we let

$$I(t) = \begin{cases} 1 & \text{if } X(t) = j, S_1 > t \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$E[\text{amount of time in state } j \text{ during 1st cycle}] = E\left[\int_0^\infty I(t)dt\right] = \int_0^\infty E[I(t)]dt = \int_0^\infty P\{X(t) = j, S_1 > t\}dt$$

So, the result follows.

Theorem . Suppose that $\{X(t), t \geq 0\}$ is a regenerative process. If $E[S_1] < \infty$, then w.p. 1

$$\lim_{t\to\infty} \frac{\text{amount of time in } j \text{ during } (0,t)}{t} = \frac{E[\text{amount of time in } j \text{ during a cycle}]}{E[\text{cycle length}]}$$

Proof.

Suppose that a reward is earned at rate 1 whenever the process is in state j. Then, it follows from the result of renewal-reward process.

Example . A tourist center makes arrangement of a tour whenever k customers are gathered. Customers arrive at the center according to a Poisson process with rate λ. Obtain the limiting probability that n customers are waiting for a tour (n=0, 1, 2, …, k-1)

(Solution) Let X(t) be the number of customers waiting at t and S_1 be the time that k customers are gathered at the first time. Then, {X(t), t≥0} will be a regenerative process. From Theorem 1,

$$\lim_{t\to\infty} P\{X(t) = n\} = \frac{E[\text{period of } n \text{ customers waiting}]}{E[S_1]}$$

But, E[period of n customers waiting] = 1/λ and $E[S_1]$=k/λ. So,

$\lim_{t\to\infty} P\{X(t) = n\} = 1/k$, k=0, 1,…,k-1.

Example . Buses arrive at a bus stop according to a Poisson process with rate μ.. Customers arrive at the bus stop according to a Poisson process with rate λ independently of buses. A bus arriving at the bus stop immediately loads all customers already arrived. What is the long-run proportion of time that k customers (k=0, 1, …) are waiting at the bus stop ?

Solution.

Every time a bus arrives will be a regenerative point. So, from Theorem 2

$$\text{Long-run proportion of time in k customers} = P_k = \frac{E[\text{amount of time with k customers}]}{E[\text{interarrival time of buses}]}$$

Let T_k be the amount of time with k customers in a cycle. Then,

$$E[T_k] = E[T_k \mid T_k = 0]P\{T_k = 0\} + E[T_k \mid T_k > 0]P\{T_k > 0\} = E[T_k \mid T_k > 0]P\{T_k > 0\}$$

T_k given that T_k >0 lasts until one more customer arrives or a bus arrives, whichever comes first. So,

$$E[T_k \mid T_k > 0] = \frac{1}{\lambda + \mu}.$$

The event $T_k > 0$ occurs when the number of customers arriving in a cycle is greater than or equal to k. This occurs when each of first k customers arrives before a bus arrives. So, from the memoryless property we have

$$P\{T_k > 0\} = \left(\frac{\lambda}{\lambda + \mu}\right)^k$$

Therefore,

$$E[T_k] = \frac{1}{\lambda + \mu}\left(\frac{\lambda}{\lambda + \mu}\right)^k$$

Hence,

$$P_k = \mu E[T_k] = \frac{\mu}{\lambda + \mu}\left(\frac{\lambda}{\lambda + \mu}\right)^k \quad \text{k=0, 1, …}$$

<u>연습문제</u>

3.1. 사건들의 발생간격이 서로 독립이고 동일한 Unif(0, 1)분포 (분포함수를 F 라 하자)를 따르는 Renewal Process 에 대하여 다음 물음에 답하라. 시간 t 까지의 발생사건수를 N(t)라 한다.

(a) F 의 n-fold convolution $F^{(n)}(t)$, n=1,2,... 를 구하라.

(b) E[N(t)]를 구하라.

(c) 시간 t 직후의 사건발생시점의 기대치를 구하라.

3.2. rate λ의 포아손과정에서 i번째 event 까지의 시간을 S_i 라 하자. 그리고 A(t) 를 시간 t 에서의 age, Y(t) 를 시간 t 에서의 excess 라 하자. 다음을 구하라.

(a) $P\{S_3 > t \mid S_2 = x\} = ?$
(b) $P\{A(t) > s\} = ?$
(c) $P\{A(t) > s, Y(t) > x\} = ?$
(d) A(t) 와 Y(t) 가 독립임을 보이라.

3.3. $\{N_1(t),\ t \geq 0\}$ 와 $\{N_2(t),\ t \geq 0\}$ 은 서로 독립인 renewal process 이며, 각각의 interarrival 분포는 F_1, F_2 와 같다. $N(t) = N_1(t) + N_2(t)$ 라 하자.

(a) $\{N(t),\ t \geq 0\}$ 는 renewal process 인가 ? 간단히 설명하라.
(b) $\lim_{t \to \infty} N(t)/t$ 를 구하라.
(c) (이 문제는 (a),(b) 와 관련없슴.)
renewal process $\{N_1(t),\ t \geq 0\}$ 의 각 event 들이 서로 독립으로 확률 p 로 count 된다고 한다. $N_C(t)$ 를 시간 t 까지 count 된 event 수라 할 때, $\{N_C(t),\ t \geq 0\}$는 renewal process 인가 ? 간단히 설명하라.
(d) $\lim N_C(t)/t$ 를 구하라.

$t \to \infty$

3.4. Taxi 승강장에 Taxi 가 rate λ 인 Poisson process 로 도착하며, 손님은 rate 가 μ 인 Poisson process 로 도착한다고 한다. 그런데 이미 한 Taxi 가 승강장에서 기다리고 있으면 다음 Taxi 는 그냥 (빈차로) 가고, 손님도 마찬가지로 어떤 한 손님이 기다리고 있으면 다른데로 간다고 한다. (즉, Taxi 가 기다리고 있다가 손님이 오면 태워가고, 손님이 기다리다가 Taxi 가 오면 타고가므로, 승강장에는 항상 비어있거나, Taxi 한대가 기다리고 있거나, 손님 한명이 기다리고있다.) 시간 0 에서는 승강장이 비어있다고 가정한다.

(a) 궁극적으로 단위시간당 몇명의 손님이 taxi 를 타고가겠는가 ?
(b) 궁극적으로 단위시간당 몇대의 taxi 가 빈차로 가겠는가 ?

3.5. n 개의 부품으로 이루어진 system 이 있는데 부품 i 는 서로 독립으로 exponential time with rate λ_i 동안 작동하다가 고장을 일으키며, 이를 수리하는데 exponential time with rate μ_i (i=1,2,...,n) 가 걸린다고한다. 한 부품이 고장나면 전체 system 이 down 되며 이것이 수리되면 다시 up 된다고한다. 한 부품이 고장나서 system 이 down 되면 다른 작동중인 부품은 그 상태로 머물러있다. 즉, 모든 부품이 작동할 때 system 이 up 상태이며, 한 부품이 고장나서 수리중일 때 system 은 down 상태이다.

(a) up period 의 평균 길이를 구하라.
(b) down period 의 평균 길이를 구하라.
(c) up-down process 가 alternating renewal process 인가 ? 그렇다면, up period 와 down period 는 독립인가 ? 간략히 설명하라.
(d) system 이 down 인 시간의 비율 (proportion of time) 을 구하라.
(e) 어떤 율 (rate) 로 system 이 down 되겠는가 ?

3.6. Renewal process 에서 A(t)를 시간 t 에서의 Age, Y(t)를 시간 t 에서의 Excess 라 할 때 다음에 대한 Reward 를 정의하고 Renewal Reward process 을 이용하여 다음을 구하라.

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{A(s)}{A(s)+Y(s)} ds$$

3.7. 포항 울릉도 선착장에서 한대의 배가 매일 일정한 시각에 한번씩 울릉도를 갔다고 돌아오는데 그 시간이 다른 모든것과 독립으로 1/2 일 또는 3/4 일 (각 확률은 0.5) 이 걸린다고한다. 한편 배를 타려는 사람들이 Poisson process with rate λ 로 도착하는데 배가 선착장에 없는 기간에 도착한 손님은 가버린다고한다.

(a) 배가 선착장에 있는 시간의 비율은 ?

(b) 배를 타려고 온 사람 중 몇 % 나 배를 탈 수 있는가 ?

(c) 승객 한 명당 A 원씩 받고 한번 출항에 C 원의 비용이 든다고 할 때 장기적으로 볼 때 단위시간당 average return 은 얼마인가 ?

3.8. 어떤 공정에서 연속적으로 부품이 생산된다. 초기에는 매 부품을 검사하다가 (전수검사; 100% inspection), k (> 1)개의 부품이 연속하여 양품이면 그 후부터는 확률 β 로 매 부품을

검 사하는 방식 (random inspection) 으로 전환하며 이 방식에서 한개의 불량품이 발견되면 다시 전수검사로 돌아가는 등 과정을 반복한다. 한 부품이 불량일 확률은 p 이다. 우리는 장기적으 로 볼 때 검사하게 되는 부품의 비율에 관심이 있다.

(a) 위의 문제를 renewal reward process 로 접근하여 단위시간당 average reward 를 계산하려 할 때, cycle 을 어떻게 정의하며 한 cycle 당 reward 를 무엇으로 정의하겠는가 ?
(b) 한 cycle 시간의 기대치를 구하라.
(c) 한 cycle 당 reward 의 기대치를 구하라.
(d) 장기적으로 볼 때 검사하게 되는 부품의 비율을 구하라.

3.9. 한명이 수리를 하는 어떤 수리점에 일감이 Poisson rate λ 로 들어온다고한다. 각 일감의 수리가 끝나면 수리공에게 보수가 주어지는데 이 보수액은 random 이며 분포 G 를 따른다고한다. 그리고 각 일감의 수리에 걸리는 시간은 보수액에 관계없이 분포 F 를 따른다고한다. 수리공이 일을 하고있을 때 들어오는 일은 받지않는다고하며, 보수액이 c 보다 큰 일감만 받는다고할 때 이 수리공의 궁극적인 단위시간당 보수는 얼마나 되겠는가 ?

3.10. 어떤 회사에 주문이 시간에 따라 들어오는데 주문간의 시간은 랜덤이며 서로 독립이고 분포 F 를 따른다고 한다. 그리고 각 주문량 역시 랜덤이며 다른 것과 독립으로 Uniform(0,100) 분포를 따른다고 한다. 이 회사는 주문량이 60 이상인 것만을 접수한다고 한다. 시간 t 까지 접수된 주문수를 N(t) 라 하자.

(a) {N(t), t≥} 는 renewal process 인가 ?
(b) 접수된 주문간의 평균시간을 구하라.
(c) 궁극적으로 볼때 단위시간당 접수된 평균 주문량 (주문수가 아니라) 을 구하라.

3.11. 어느 고속도로에 속도위반차량을 단속하는 교통순경이 서있는데 차량이 분당 λ대의 포아손과정으로 지나간다고한다. 차량들의 속도는 시속 a km 에서 시속 b km (a < b) 사이의 uniform 분포를 따른다. 속도제한은 시속 100 km 이다. 이 순경은 시속 v (>100) km 이상으로 달리는 차량만 단속한다. 시속 V km 로 달리다 잡히면 벌금을 c(V-100) 원을 내야한다 (V > 100 인 경우). 한 차량을 세우고 벌금을 물리는데 걸리는 시간은 다른 것과 독립으로 분포 G를 따른다. 한 차량을 단속중에 속도위반으로 통과하는 다른 차량은 잡을 수 없다.

(a) 단속한 차량의 수를 count하는 과정이 renewal process임을 간단히 설명하라.
(b) 한 차량의 단속을 막 끝내고 다음 단속차량까지 지나가는 차량수의 분포는 ?
(c) (a)의 renewal process의 renewal cycle의 기대치를 구하라.
(d) long-run에서 단위시간당 평균단속차량수를 구하라.
(e) long-run에서 단위시간당 평균벌금액을 구하라.

3.12. 손님들이 어떤 지하철역에 rate λ 의 포아손 과정으로 도착한다. 시간 t 까지 도착한 손님수를 N(t) 라 하자. 지하철은 시각 T, 2T, ... 등에 떠난다. (여기서 T 는 상수) 도착한 손님은 다음에 오는 지하철에 모두 탄다. i 번째 손님이 지하철 탈 때까지 기다리는 시간을 W_i 라고 하자. 그리고 시간 t 까지의 도착한 모든 손님들의 총 기다리는 시간을 다음과 같이 정의한다.

$$W(t) = \sum_{i=1}^{N(t)} W_i \qquad \text{모든 } t \geq 0$$

(a) E[W(T)] (즉, 첫 지하철에 탈 때까지 기다린 손님들의 총시간의 기대치) 를 구하라.

(b) $\lim_{t->\infty} \frac{W(t)}{t}$ 를 구하라.

(c) 지하철의 떠나는 간격이 상수 T 가 아니라 기대치 E[X] 를 갖는 iid 확률변수라 할 때 물음 (a), (b) 에 답하라.

3.13. 갑이라는 학생은 도서관에 rate λ 의 포아손과정으로 방문한다. n 번째 방문에서 과거와 독립적으로 A_n 권의 책을 대출한다. 단, P{ A_n =i}=a_i, i=0,1,2,... 또한 매 방문시 이미 대출중인 책을 반납할 확률 (권당)이 p 이다. 갑이 시간 t 까지 대출한 총 책의 수 (반납을 고려치 않은) 를 C(t) 라 하자.

(a) {C(t), t≥0} 는 어떤 Process 인가 ?

(b) $\lim_{t->\infty} \frac{C(t)}{t}$ 를 구하라.

(c) 갑이 대출한 한 권의 책을 갖고 있는 기간의 분포를 구하라.

(d) 책의 대출기간은 d (상수) 라고 한다. 대출기간이 지나면 지난기간 단위시간당 c 원의 벌금을 내야한다. long-run 으로 볼 때, 갑이 내야하는 단위시간당 벌금은 ?
(Hint: 단위시간당 벌금은 단위시간당 권수에 권당 벌금기대치의 곱으로 산출할 수 있다.)

3.14. 부품 1 과 부품 2 가 병렬로 연결된 시스템이 있다. 부품 1 과 부품 2 의 수명은 독립이며 각각 분포 F_1, F_2 를 따른다고 한다. 이 시스템은 두 부품 중 적어도 하나가 작동하면 동작하며 모두 고장나면 시스템이 고장난다. 시스템이 고장나면 즉시 두 부품을 새것 (동일한 수명분포를 갖는 것)들로 각각 교체한다. 시간 0 에 두 부품은 새것이라 하자.

(a) 시간 t 까지의 시스템고장수를 N(t)로 나타낼 때, {N(t), t≥0}를 Renewal Process 라고 할 수 있는가 ? 두부품의 수명이 동일한 지수분포를 따른다고 할 때 {N(t), t≥0} 를 Poisson Processs 라 할 수 있는가 ? 간단히 설명하라.
(b) 한 시스템수명주기에서 두 부품이 모두 작동하고 있는 기간의 기대치를 구하라. (F_1, F_2 으로 표현)
(c) 두부품이 모두 작동하고 있는 기간동안에는 단위시간당 생산량이 r_2 이며, 어느 한 부품만 작동하고 있는 기간에는 단위시간당 생산량이 r_1이라 한다. Long-run 으로 볼 때, 이 시스템의 단위시간당 평균생산량은 얼마인가 ? (F_1, F_2 으로 표현)

3.15. 부품 1, 부품 2의 두 부품으로 이루어진 시스템이 있다. 각 부품의 수명은 서로 독립이며, 부품 i 의 수명은 rate λ_i (i=1,2) 의 지수분포를 따른다. 그런데 부품 1 은 시스템의 수명에 치명적 영향을 주므로 부품 1이 고장이 나면 곧 수리에 착수하나, 부품 2는 치명적이 아니므로 고장이 나더라도 다음번 부품 1의 고장까지 미루어둔다 한다. 즉, 부품 2가 부품 1 이전에 고장난 경우만 부품 1 수리시 부품 2를 동시에 수리한다. 각 부품은 수리후에는

새것과 동일하게 작동한다고 한다. 수리에 걸리는 시간은 부품 1만 하던 두 부품을 동시에 하던 고장시간과는 독립으로 평균 $1/\mu$ 의 지수분포를 따른다. 부품 1 수리기간 동안에는 부품 2가 살아있다고 하여도 동작을 멈추므로 고장이 발생할 수 없다. 부품 i 의 수리비용은 c_i (i=1,2) 이다.

(a) renewal process 로 분석하고자 할 때 적절한 renewal cycle 을 정의하라.
(b) renewal cycle 의 기대치를 구하라.
(c) cycle 내에서 부품 2 역시 수리될 확률을 구하라.
(d) long-run average cost (per unit time) 를 구하라.

3.16. 어떤 부품의 고장 여부를 test 하는데 test 간의 시간 X_n (n=1,2,...) 은 iid random variable 이며 분포함수가 F(t), $E[X_n] = \mu$ 라한다. test 에서 부품이 고장인 것을 발견하면 즉시 새것으로 교체한다고 한다. 부품의 수명은 평균이 $1/\lambda$ 인 iid exponential 분포를 따른다. 한번 test 하는데 비용은 C_t, 부품 교체비용은 C_r 이며, test 사이 시점에서 고장났을 경우에는 고장나있은 기간 (다음 test 까지의 기간) 의 k 배의 비용이 든다고 한다.

(a) test 에서 작동중인 부품의 잔여수명의 기대치는 ?
(b) 다음을 증명하라.
  (i) test 간 기간중 부품의 작동시간의 기대치가 아래와 같다.

$$\int_0^\infty e^{-\lambda t} F^c(t)\,dt$$

  (ii) 부품이 test 간 기간중 작동할 확률은 아래와 같다.

$$\int_0^\infty F(t)\,\lambda e^{-\lambda t}\,dt$$

(c) (b) 의 결과를 이용하여 위의 교체 방안의 단위시간당 비용을 구하라.

3.17. 어떤 설비가 random 시간동안 작동하다가 고장이 나면 또 다른 random 시간동안 수리를 요하며 수리가 끝나면 다시 작동한다고 한다. j번째 작동시간 및 j번째 수리시간을 각각 X_{uj}, X_{dj} 라 하자. $X_j = X_{uj} + X_{dj}$ 로 정의하며, X_j 는 iid 이다. 작동시간은 분포 F 를 따르며, 수리시간은 k 종류의 서로 배타적인 고장원인에 따라 다른 분포를 따르는데 i 종류의 고장일 때 수리시간은 분포G_i 를 따르며, 비용이 c_i 든다. 그리고 고장원인이 i 일 확률은 p_i ($\sum_{i=1}^{k} p_i = 1$) 이다.

(a) renewal cycle 을 정의하고 mean cycle time 을 구하라. (이 시간은 finite 하다.)
(b) i 종류의 고장에 의한 수리가 진행중일 시간비율을 구하라.
(c) 단위시간당 드는 비용을 구하라.
(d) 어떤 임의의 시점에서 고장원인 i 에 의한 수리가 진행중이라 할 때, 잔여 수리시간의 분포를 구하라.

3.18. interarrival time 의 분포가 iid F 인 renewal process에서 시간 t까지의 renewal 수를 N(t)라 할 때, $E[N(t)^2]$ 에 대한 renewal equation을 유도하고 그 해를 구하라.

3.19. interarrival time 의 분포가 iid F 인 renewal process에서 시간 t 직후의 renewal 시점 $S_{N(t)+1}$ 에

대하여 renewal equation을 유도하고 그 해를 구하라. 결과가 Wald's Equation을 사용한 것과 동일함을 확인하라.

3.20. interarrival time 이 iid F 인 renewal process에서 시간 t 에서의 age 를 A(t) 로 나타낼 때 다음을 구하라.

(a) E[A(t)] 에 관한 renewal equation 을 유도하고 그 해를 구하라.
(b) $\lim_{t\to\infty} E[A(t)] = ?$

3.21. interarrival time 의 분포가 iid F 인 renewal process에서 시간 t에서의 Spread를 X(t)라 하자.

(a) E[X(t)]의 renewal equation을 유도하고 그 해를 구하라.
(b) $\lim_{t\to\infty} E[X(t)]$ 를 구하라. Average spread와 동일한가 ?

3.22. interarrival time 이 iid F 인 renewal process에서 시간 t 에서의 excess 를 Y(t) 로 나타낼 때 다음을 구하라.

(a) $P\{Y(t) \le y\}$ 에 관한 renewal equation 을 유도하고 그 해를 구하라.
(b) $\lim_{t\to\infty} P\{Y(t) \le y\} = ?$ (극한치 존재 가정)

3.23. Renewal Process 에서 Y(t) 를 시간 t 에서의 Excess 라 하자. Interarrival time X_i's 은 분포 F (pdf: f) 를 따르며 $E[X^3] < \infty$ 이다.

(a) $E[(Y(t))^2]$ 에 대한 renewal equation 을 유도하라.
(b) $E[(Y(t))^2]$ 의 pointwise limit 을 구하라. 이 극한치가 존재하기 위하여 어떤 조건들을 check 하여야 하는가 ?

3.24. 어떤 시스템의 상태는 1 -> 2 -> 3 ->...-> n -> 1 ... 등으로 n 개의 상태를 순환적으로 취하는데, 상태 i 에 머무르는 시간은 기대치 $\mu_i < \infty$ 를 가진 분포 F_i 를 따른다. 그리고 시간 0 에 상태 i 를 막 방문하였을 때 시간 t 에 상태 j 에 있을 확률을 $P_{ij}(t)$ 라 하자.

(a) P_j 를 long-run 에서 상태 j 에 있을 시간의 비율 (fraction of time)이라 할 때 이를 구하라.

(b) 다음이 성립함을 증명하라.

$$P_{jj}(t) = F_j^c(t) + \int_0^t F_j^c(t-x)dm(x)$$

윗식에서 m(x)는 정의된 기호들로 어떻게 표현되는가 ?

(c) 어떤 조건에서 극한치 $\lim_{t\to\infty} P_{jj}(t)$ 가 존재하는가 ? 이 때 극한치를 구하라. 결과를 (a)와 비교하라.

3.25. "up", "down" 으로 이루어지는 alternating renewal process 에서 up time 의 분포는 F_u, down time 은 F_d 를 따른다. (up + down) cycle 은 분포 F 를 따르며 그 평균은 μ_F $(< \infty)$ 라고 한다. 시간 0 에서 up period 가 시작된다고한다. 임의의 시간 t 에 대해 K_t 를 다음과

같이 정의한다.
K_t = 시간 t 에서 진행중인 up period 의 남은 기간 (t 에서 up 일때)
= 0 (t 에서 down 일때)

(a) 어떤 정해진 k ≥ 0 에 대해 $H(t) = P\{K_t > k\}$ (t > 0) 으로 정의할 때 다음을 증명하라.

$$H(t) = F_u^c(t+k) + \int_0^t F_u^c(t+k-x)\,dm(x)$$

(b) 다음의 극한치를 구하라. 어떤 조건에서 이 극한치가 존재하는가 ?

$$\lim_{t\to\infty} H(t)$$

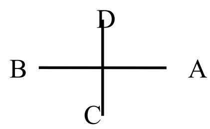
(c) (a), (b) 와 관계없이 renewal reward process 를 이용하여 다음의 극한치를 구하라.

$$\lim_{t\to\infty}\int_0^t H(x)dx/t$$

3.26. 어떤 한 사람이 버스 승강장에 random 으로 도착한다고 하자. 그리고 버스는 시간당 3 대의 빈도수로 도착한다고 하자. 이 때 버스의 도착간격이 다음과 같은 분포를 갖을 때, 이 사람이 버스를 기다려야하는 시간의 분포를 구하라.

(a) constant
(b) exponential
(c) 0 또는 60 분 (즉, 3 대의 버스가 같이 도착하는 경우) 복잡한 서울 시내에서는 이런 현상을 많이 볼수있는데 어떤 이유에서인가?

3.27. 아래 그림과 같은 신호등이 있는 교차로가 있다. 네군데 모두 직진과 좌회전 동시신호체계인데, A->B/C 방향과 B->A/D 방향의 초록불 지속시간은 다른 것과 독립적으로 분포 F (기대치=μ_F, 분산=σ_F^2)를 따르며, C->D/B 방향과 D->C/A 방향의 초록불 지속시간은 다른 것과 독립적으로 분포 G(기대치=μ_G, 분산=σ_G^2)를 따른다. 초록불이 켜지는 순서는 A->B/C 방향, C->D/B 방향, B->A/D 방향, D->C/A 방향, 다시 A->B/C 방향 등의 순이다.



(a) 임의의 시각에 C 에 도착한 차량이 즉시 좌회전할 수 있을 확률을 구하라.
(초록불에서 통과시간 무시)
(b) 임의의 시각에 C 에 도착한 차량이 좌회전하기 위하여 신호를 기다리는 평균시간을 구하라.

3.28. inter-arrival time X_i 가 분포 F (density f) 를 갖는 renewal process 를 고려하자. 단, $0 < E[X_i] = \mu < \infty$. 이 renewal process 의 age 와 excess 를 각각 A(t), Y(t) 라 할 때 새로운 process {D(t), t≥0} 을 다음과 같이 정의한다. $D(t) = \min[A(t), Y(t)]$

(a) $X_1 = 2$, $X_2 = 4$, $X_3 = 1$ 일 때 시간 [0, 7] 에서의 {D(t), t≥0} 의 sample path 를 그리라.
(b) {D(t), t≥0} 는 regenerative process 인가 ? 간단히 설명하라.
(c) $K(t) = P\{D(t) > z\}$ 라 할 때, $K(t) = Q(t) + Q*m(t)$ 가 된다고 한다. Q(t) 를 구하라. (여기서 m(t) 는 renewal function 임)
(d) 어떤 조건에서 $\lim_{t\to\infty} K(t)$ 가 존재하는가 ? 이 조건이 만족한다고 할 때 이 값을 구하라.

2002-09-12 C.H. JUN