Ordinary Differential Equations

Exact Equations and Integrating Factor

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Exact Equations & Integrating Factors

- Consider a first order ODE of the form
$$M(x,y) + N(x,y)y' = 0$$
- Suppose there is a function ψ such that
$$\psi_x(x,y) = M(x,y), \psi_y(x,y) = N(x,y)$$
and such that $\psi(x,y) = c$ defines $y = \phi(x)$ implicitly. Then
$$M(x,y) + N(x,y)y' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi(x,\phi(x))$$
and hence the original ODE becomes
$$\frac{d}{dx}\psi\big(x,\phi(x)\big) = 0$$
- Thus $\psi(x,y) = c$ defines a solution implicitly.
- In this case, the ODE is said to be **exact**.

Theorem 1

Suppose an ODE can be written in the form

$$M(x,y) + N(x,y)y' = 0 \qquad (1)$$

where the functions M, N, M_y and N_x are all continuous in the rectangular region $\mathcal{R}$: $(x,y) \in (\alpha,\beta) \times (\gamma,\delta)$. Then Eq. (1) is an **exact** differential equation iff

$$M_y(x,y) = N_x(x,y), \forall(x,y) \in \mathcal{R} \qquad (2)$$

That is, there exists a function ψ satisfying the conditions

$$\psi_x(x,y) = M(x,y), \qquad \psi_y(x,y) = N(x,y) \qquad (3)$$

iff M and N satisfy Eq. (2).

Example 1: Exact Equation

Consider the following differential equation.

$$\frac{dy}{dx} = -\frac{x + 4y}{4x - y}$$

Then $M(x, y) = x + 4y,\ N(x, y) = 4x - y$
and hence

$$M_y(x, y) = 4 = N_x(x, y) \Rightarrow \text{ODE is exact}$$

From Theorem 1, there exist $\psi(x, y)$ such that

$$\psi_x(x, y) = x + 4y, \qquad \psi_y(x, y) = 4x - y$$

Thus

$$\psi(x, y) = \int \psi_x(x, y)dx = \int (x + 4y)dx = \frac{1}{2}x^2 + 4xy + C(y)$$

Example 1: Solution (2 of 4)

We have $\psi_x(x,y) = x + 4y, \quad \psi_y(x,y) = 4x - y$
and

$$\psi(x,y) = \int \psi_x(x,y)dx = \int (x+4y)dx = \frac{1}{2}x^2 + 4xy + C(y)$$

It follows that

$$\psi_y(x,y) = 4x - y = 4x - C'(y) \Rightarrow C'(y) = -y \Rightarrow C(y) = -\frac{1}{2}y^2 + k$$

Thus

$$\psi(x,y) = \frac{1}{2}x^2 + 4xy - \frac{1}{2}y^2 + k$$

By Theorem 1, the solution is given implicitly by

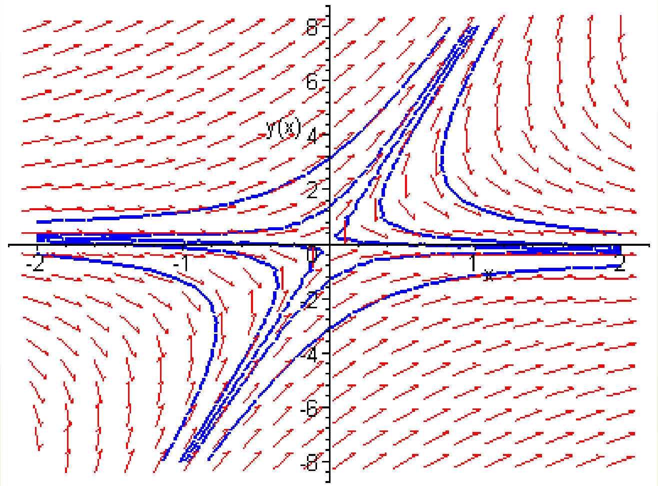
$$x^2 + 8xy - y^2 = c$$

Example 1:
Direction Field and Solution Curves

Our differential equation and solutions are given by

$$\frac{dy}{dx} = -\frac{x+4y}{4x-y} \Leftrightarrow (x+4y) + (4x-y)y' = 0 \Rightarrow x^2 + 8xy - y^2 = 0$$

A graph of the direction field for this differential equation, along with several solution curves, is given below.



Example 2: Exact Equation (1 of 3)

Consider the following differential equation.

$$(y\cos x + 2xe^y) + (\sin x + x^2 e^y - 1)y' = 0.$$

Then $M(x,y) = y\cos x + 2xe^y, N(x,y) = \sin x + x^2 e^y - 1$
and hence

$$M_y(x,y) = \cos x + 2xe^y = N_x(x,y) \Rightarrow \text{ODE is Exact}$$

From Theorem 1,

$$\psi_x(x,y) = M = y\cos x + 2xe^y, \psi_y(x,y) = N = \sin x + x^2 e^y - 1$$

Thus

$$\psi(x,y) = \int \psi_x(x,y)dx = \int (y\cos x + 2xe^y)dx = y\sin x + x^2 e^y + C(y)$$

Example 2: Solution (2 of 3)

From this

$$\psi(x,y) = \int \psi_x(x,y)dx = \int (y\cos x + 2xe^y)dx = y\sin x + x^2e^y + C(y)$$

It follows that

$$\psi_y(x,y) = \sin x + x^2e^y - 1 = \sin x + x^2e^y + C'(y)$$

$$\Rightarrow C'(y) = -1 \Rightarrow C(y) = -y + k$$

Thus

$$\psi(x,y) = y\sin x + x^2e^y - y + k$$

Theorem 1, the solution is given implicitly by

$$y\sin x + x^2e^y - y = c.$$

Example 2:
Direction Field and Solution Curves

Our differential equation and solutions are given by

$$(y\cos x + 2xe^{y}) + (\sin x + x^{2}e^{y} - 1)y' = 0$$

$$y\sin x + x^{2}e^{y} - y = c.$$

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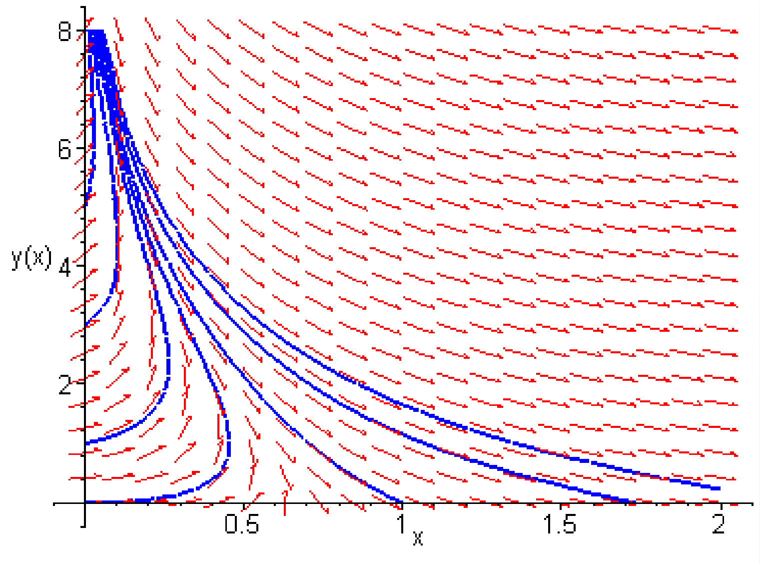

Example 3: Non-Exact Equation

Consider the following differential equation.

$$(3xy + y^2) + (2xy + x^3)y' = 0$$

Then

$$M(x,y) = 3xy + y^2, \qquad N(x,y) = 2xy + x^3$$

and hence

$$M_y(x,y) = 3x + 2y \neq 2y + 3x^2 = N_x(x,y) \Rightarrow \text{ODE is not exact}$$

To show that our differential equation cannot be solved by this method, let us seek a function ψ such that

$$\psi_x(x,y) = M = 3xy + y^2, \qquad \psi_y(x,y) = N = 2xy + x^3$$

Thus

$$\psi(x,y) = \int \psi_x(x,y)dx = \int (3xy + y^2)dx = \frac{3}{2}x^2y + xy^2 + C(y)$$

Example 3: Non-Exact Equation

It follows that

$$\psi_y(x,y) = 2xy + x^3 = \frac{3}{2}x^2 + 2xy + C'(y)$$

$$\Rightarrow C'(y) \overset{?}{=} x^3 - \frac{3}{2}x^2 \Rightarrow C(y) \overset{??}{=} x^3 y - \frac{3}{2}x^2 y + k$$

Thus there is no such function ψ. However, if we (incorrectly) proceed as before, we obtain

$$x^3 y + xy^2 = c$$

as our implicitly defined y, which is not a solution of ODE.

Integrating Factors

It is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor $\mu(x, y)$:

$$M(x, y) + N(x, y)y' = 0$$

$$\mu(x, y)M(x, y) + \mu(x, y)N(x, y)y' = 0$$

For this equation to be exact, we need

$$(\mu M)_y = (\mu N)_x \Leftrightarrow M\mu_y - N\mu_x + \left(M_y - N_x\right)\mu = 0$$

This partial differential equation may be difficult to solve. If μ is a function of x alone, then $\mu_y = 0$ and hence we solve

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$

provided right side is a function of x only.

Similarly if μ is a function of y alone. $\Rightarrow \frac{d\mu}{dy} = -\frac{M_y - N_x}{M}\mu$

Example 4: Non-Exact Equation

Consider the following non-exact differential equation.

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

Seeking an integrating factor, we solve the linear equation

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \Leftrightarrow \frac{d\mu}{dx} = \frac{\mu}{x} \Rightarrow \mu(x) = x$$

Multiplying our differential equation by μ, we obtain the exact equation

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0,$$

which has its solutions given implicitly by

$$x^3y + \frac{1}{2}x^2y^2 = c.$$

Integrating Factor

For $\mu = f(u)$, where $u = g(x, y)$ we have

$$(\mu M)_y = (\mu N)_x \Leftrightarrow M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

That becomes

$$M\frac{d\mu}{du}u_y - N\frac{d\mu}{du}u_x + (M_y - N_x)\mu = 0$$

or

$$\frac{d\mu}{\mu} = \frac{N_x - M_y}{Mu_y - Nu_x}du$$

EXAMPLE 4: INTEGRATING FACTOR $\boldsymbol{\mu = f(u)}$

Solve the equation $y\,dx + (x^2 + y^2 - x)\,dy = 0$ using the integrating factor as a function of $x^2 + y^2$

Solution **:**

First of all we check this equation for exactness:

$$\frac{\partial M}{\partial y} = M_y = 1, \qquad \frac{\partial N}{\partial x} = N_x = 2x - 1$$

The partial derivatives of $M_y \neq N_x$ are not equal to each other. Therefore, this equation is not exact.

Now we try to use the integrating factor in the form $u = x^2 + y^2$.

Here we have $\frac{\partial u}{\partial x} = 2x, \;\; \text{and} \; \frac{\partial u}{\partial y} = 2y$

Then
$$\frac{1}{\mu}\frac{d\mu}{du}=\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N\frac{\partial u}{\partial x}-M\frac{\partial u}{\partial y}}=\frac{1-(2x-1)}{(x^2+y^2-x)2x-(y)2y}$$
$$=\frac{2-2x}{(x^2+y^2)(2x-2)}=\frac{1}{-(x^2+y^2)}=-\frac{1}{u}$$
$$\frac{1}{\mu}\,d\mu=-\frac{1}{u}du$$
$$\mu=u^{-1}=\frac{1}{(x^2+y^2)}$$

By the function $\mu(x,y)$ we can convert the original differential equation into exact
$$\frac{y}{(x^2+y^2)}dx+\frac{x^2+y^2-x}{(x^2+y^2)}dy=0$$
where $\overline{M}(x,y)=\frac{y}{(x^2+y^2)}$ and $\overline{N}(x,y)=\frac{x^2+y^2-x}{(x^2+y^2)}=1-\frac{x}{(x^2+y^2)}$

Integrate the first equation with respect to the variable x (considering y as a constant):
$$\psi(x,y)=\int\overline{M}\,(x,y)dx=\int\frac{y}{(x^2+y^2)}dx=\arctan\frac{x}{y}+C(y)$$

Substitute this in the first equation system to get:

$$\frac{\partial u(x,y)}{\partial y} = \frac{-x}{(x^2+y^2)} + C'(y) = \overline{N}(x,y)$$

We have $C'(y) = 1$ then C$(y) = y + k$

Hence, the general solution of the given differential equation is written in the form:

$$\arctan\frac{x}{y} + y + k = 0$$

where k is any real number.

Exercises

Find the integrating factor, then solve the following ODEs:

1. $(x + y)dx + dy = 0$
2. $2xy(1 + y^2)dx - (1 + x^2 + x^2y^2)dy = 0$
3. $y + xy^2 + (x - x^2y)y' = 0$ using an integrating factor as a function of xy.