Ordinary Differential Equations

Clairaut Equation & D'Alembert/Lagrange Equation

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LAGRANGE/D'Alembert EQUATION

- An implicit differential equation of type $y = f(x, y')$ of the following form

$$y = x.\varphi(y') + \psi(y')$$

where $\varphi(y')$ and $\psi(y')$ are known functions differentiable on a certain interval, is called the **Lagrange equation.**

- By setting $y' = p$ and differentiating with respect to x, we get the general solution of the equation in parametric form:

$$\begin{cases} x = f(p, C) \\ y = f(p, C)\varphi(p) + \psi(p) \end{cases}$$

provided that $\varphi(p) - p \neq 0$, where p is a parameter.

- Lagrange equation may also have a <u>singular solution</u>
if the condition $\varphi(p) - p \neq 0$ is failed (or if $\varphi(p) - p = 0$).
singular solution is given by the expression: $y = x.\varphi(p^*) + \psi(p^*)$
where p^* is the root of the equation $\varphi(p) - p = 0$

Example 1

Find the general and singular solutions of the differential equation

$$y = 2xy' - 3(y')^2$$

<u>*Solution.*</u>

Here we see that we deal with a Lagrange equation. We will solve it using the method of differentiation.

- Denote $y' = p$, so the equation is written in the form: $y = 2xp - 3p^2$
- Differentiate both sides with respect to x, we have:

$$\frac{dy}{dx} = 2p + (2x - 6p)\frac{dp}{dx}$$

$$\Leftrightarrow p = 2p + (2x - 6p)\frac{dp}{dx}$$

$$\Leftrightarrow \frac{dx}{dp} + \frac{2}{p}x - 6 = 0$$

As it can be seen, we obtain a <u>linear equation</u> for the function $x(p)$.

- The integrating factor is $\mu(p) = \exp\int \frac{2}{p}\,dp = \exp \ln|p|^2 = p^2$
- The general solution of the linear equation is given by

$$p^2.x(p) = \int p^2.6dp + C$$

$$x(p) = 2p + \frac{C}{p^2}$$

- Substituting this expression for x into the Lagrange equation, we obtain:

$$y = 2\left(2p + \frac{C}{p^2}\right)p - 3p^2 = p^2 + \frac{2C}{p}$$

- Thus, the general solution in parametric form is defined by the system of equations: $\begin{cases} x(p) = 2p + \frac{C}{p^2} \\ y(p) = p^2 + \frac{2C}{p} \end{cases}$

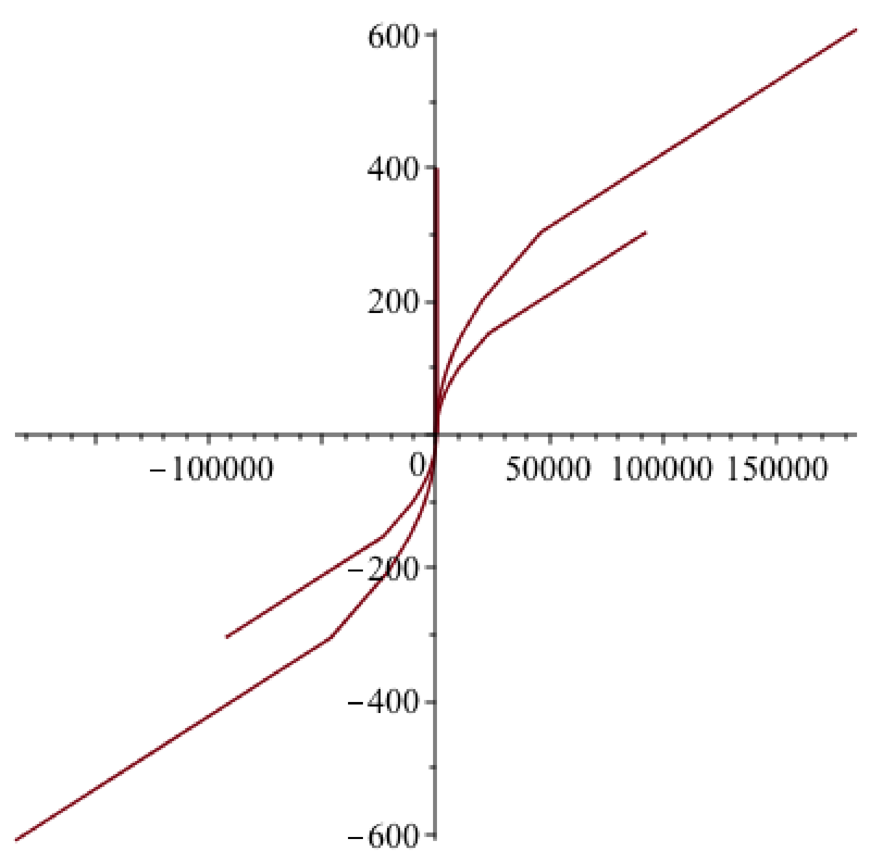
- Besides, the Lagrange equation can have a singular solution. Solving the equation $\varphi(p) - p = 0,$

we find the root:

$$2p - p = 0, \qquad \Longrightarrow p = 0$$

- Hence, the singular solution is expressed by the linear function:

$$y = \varphi(0)x + \psi(0) = 0$$



Example 2

Find the general and singular solutions of the equation

$$2y - 4xy' - \ln y' = 0$$

<u>*Solution.*</u>

Here we have a Lagrange equation. By setting $y' = p$, we can write:
$2y = 4xp + \ln p$

Differentiate both sides by x, we have:

$$2\frac{dy}{dx} = 4p + \left(4x + \frac{1}{p}\right)\frac{dp}{dx}$$

$$2p = 4p + \left(4x + \frac{1}{p}\right)\frac{dp}{dx}$$

$$\frac{dx}{dp} + \frac{2}{p}x = \frac{1}{2p^2}$$

Thus, we get a <u>linear differential equation</u> for the function $x(p)$.

using the integrating factor : $\mu(p) = \exp\left(\int \frac{2}{p} dp\right) = \exp(\ln|p|^2) = p^2$

The function $x(p)$ is defined by

$$x(p)p^2 = \int p^2 \left(-\frac{1}{2p^2}\right) dp + C$$

$$x(p) = -\frac{1}{2p} + \frac{C}{p^2}$$

Substituting this into the original equation, $2y = 4xp + \ln p$

$$\Leftrightarrow 2y = 4\left(-\frac{1}{2p} + \frac{C}{p^2}\right)p + \ln p$$

$$\Leftrightarrow y = \frac{2C}{p} - 1 + \frac{\ln p}{2}$$

Hence, the general solution in parametric form is written as follows:

$$\begin{cases} x(p) = \dfrac{C}{p^2} - \dfrac{1}{2p} \\ y(p) = \dfrac{2C}{p} - 1 + \dfrac{\ln p}{2} \end{cases}$$

To find the singular solution, we solve the equation:

$$\varphi(p) - p = 0, \Longrightarrow 2p - p = 0, \Longrightarrow p = 0$$

It follows from this that $y = C$. We can make direct substitution to make sure that the constant C is equal to zero.

Thus, the differential equation has the singular solution $y = 0$. We have already met with this solution above when we divided the equation by p.

Clairaut Equation

- If Lagrange Equation $y = x.\varphi(y') + \psi(y')$ with $\varphi(y') = \mathrm{y}'$, then we have

$$y = x.y' + \psi(y')$$

This is called **Clairaut Equation**.

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It is solved in the same way by introducing a parameter $\mathrm{y}' = p$ and differentiating both sides of the equation to have: $\{x + \psi'(p)\}\frac{dp}{dx} = 0.$

From $\frac{dp}{dx} = 0$ we obtain $y = \mathrm{C}, \mathrm{C}$ arbitrary constant. **The general solution** is given by $y = Cx + \psi(C)$.

- Clairaut equation may have a **singular equation** that is given by:

$$\begin{cases} x = -\psi'(p) \\ y = xp + \psi(p) \end{cases}$$

where p is a parameter.

Example 3

Find the general and singular solutions of the differential equation $y = xy' + (y')^2$.

<u>*Solution.*</u>

This is a Clairaut equation.

By setting $y' = p$, we write it in the form $\qquad y = xp + p^2$

Differentiating in x, we have

$$\frac{dy}{dx} = p + (x + 2p)\frac{dp}{dx}$$

$$p = p + (x + 2p)\frac{dp}{dx}$$

$$0 = (x + 2p)\frac{dp}{dx}$$

$$0 = (x + 2p)dp$$

By equating the first factor to zero, we have $dp = 0\,, \Longrightarrow p = C$

Now we substitute this into the differential equation to have: $y = Cx + C^2$

Thus, we obtain the **general solution** of the Clairaut equation, which is an one-parameter *family of straight lines*.

- By equating the second term to zero we find that $x + 2p = 0, \Rightarrow x = -2p$
- This gives us **the singular solution** of the differential equation in parametric form:

$$\begin{cases} x = -2p \\ y = xp + p^2 \end{cases}$$

- By eliminating p from this system, we get the equation of the integral curve:

$$p = -\frac{x}{2}, \qquad \Rightarrow y = x\left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2$$

$$y = -\frac{x^2}{4}$$

From geometric point of view, the curve $y = -\frac{x^2}{4}$ is the envelope of the family of straight lines defined by the general solution (see Figure 1).

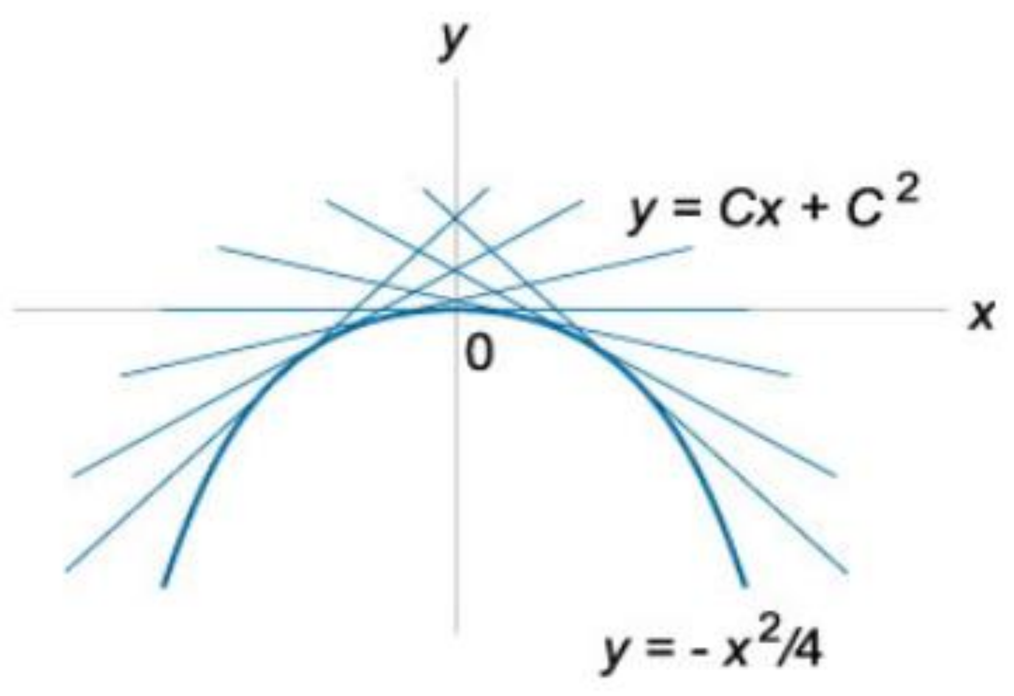


Figure 1.

Example 4

Find the general and singular solutions of the ODE $y = xy' + \sqrt{(y')^2 + 1}$

<u>*Solution.*</u>

As it can be seen, this is a Clairaut equation. Introduce the parameter $y' = p$, we have : $y = xp + \sqrt{p^2 + 1}$

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = p + \left(x + \frac{p}{\sqrt{p^2+1}}\right)\frac{dp}{dx}$$

$$p = p + \left(x + \frac{p}{\sqrt{p^2+1}}\right)\frac{dp}{dx}$$

$$\left(x + \frac{p}{\sqrt{p^2+1}}\right)dp = 0$$

Consider the case $dp = 0$, then $p = C$.

Substituting this in the equation, we find the general solution: $y = Cx + \sqrt{C^2 + 1}$

Graphically, this solution corresponds to the family of one-parameter straight lines.

- The second case is described by the equation $x = -\frac{p}{\sqrt{p^2+1}}$.

  Find the corresponding parametric expression for y:

$$y = xp + \sqrt{p^2 + 1}$$

$$y = -\frac{p^2}{\sqrt{p^2 + 1}} + \sqrt{p^2 + 1}$$

$$y = \frac{1}{\sqrt{p^2 + 1}}$$

- The parameter p can be eliminated from the formulas for x and y.

$$x^2 + y^2 = \left(-\frac{p}{\sqrt{p^2+1}}\right)^2 + \left(\frac{1}{\sqrt{p^2+1}}\right)^2 = 1$$

The last expression is the equation of the circle with radius 1 and centered at the origin. Thus, the singular solution is represented by the unit circle on the xy-plane, which is the envelope of the family of the straight lines (Figure 2).

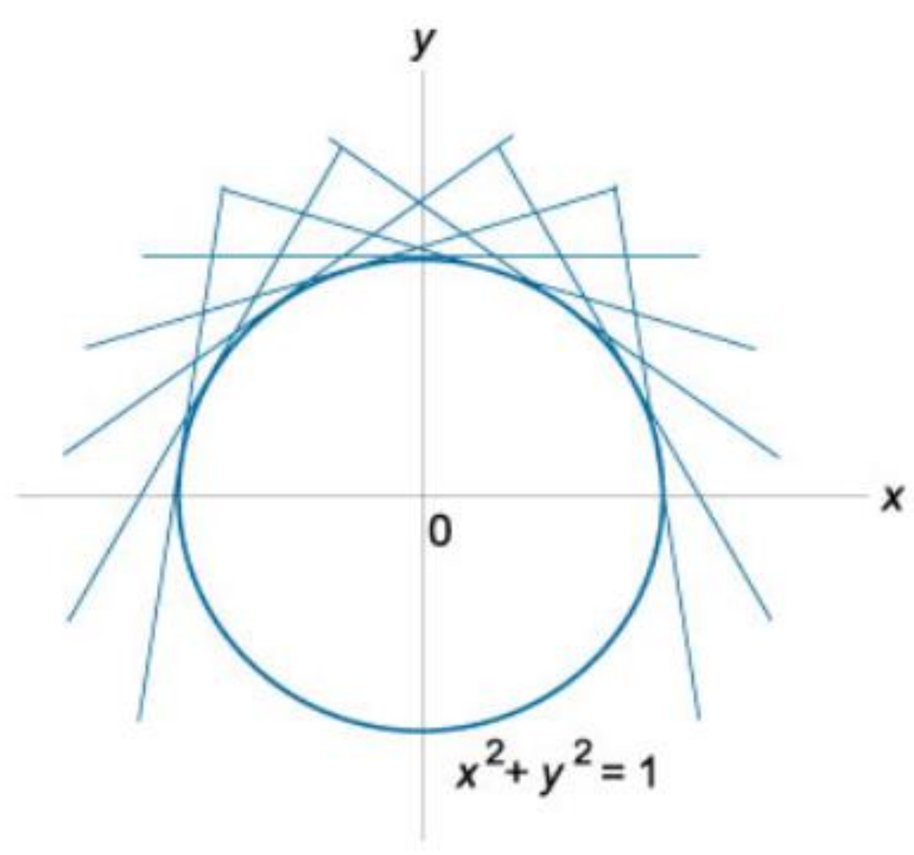


Figure 2.

Exercises

Find the general solution and singular solution of the following equations and sketch the solutions using Maple

1. $y = px - 2p^2$
2. $xp^2 - 2yp + 4x = 0.$

Clairaut Equation

Clairaut equation is special case of implicit equation **type 2:** $\boldsymbol{y = f(x, y')}$.

The Clairaut equation has the form:

$$y = xy' + \psi(y')$$

where $\psi(y')$ is a nonlinear differentiable function.

By setting $y' = p$ and differentiating with respect to x, we get the general solution of the equation in parametric form:

$$y = Cx + \psi(C),$$

where C is an arbitrary constant.

The Clairaut equation may have a singular solution that is expressed parametrically in the form:

$$\begin{cases} x = -\psi(p) \\ y = xp + \psi(p) \end{cases}$$

where p is a parameter.

example

- Find the general and singular solutions of the differential equation $y = xy' + (y')^2$.

Solution:

By setting $y' = p$, we write it in the form $y = xp + p^2$.

Differentiating in x,, we have

- dy=xdp+pdx+2pdp.
- Replace dy with pdx to obtain:
- pdx
- =xdp+pdx+2pdp,⇒dp(x+2p)=0.
- By equating the first factor to zero, we have
- dp=0,⇒p=C.

Definition and Methods of Solution

An equation of type $\boldsymbol{F(x, y, y') = 0}$
where F is a continuous function, is called the **first order implicit differential equation**.

The main techniques for solving an implicit differential equation is the method of introducing a parameter. Below we show how this method works to find the general solution for some most important particular cases of implicit differential equations.

There are five types in Implicit Differential Equations.

- $(y' - F_1)(y' - F_2) \dots (y' - F_n) = 0$
- $y = f(x, y')$
- $x = f(y, y')$
- $y = f(y')$
- $x = f(y')$

Type 2: Implicit Differential Equation of Type $y = f(x, y')$

Let the parameter $p = y' = \frac{dy}{dx}$ and differentiate the equation

$y = f(x, y') = f(x, p)$ with respect to x to have:

$$\frac{dy}{dx} = \frac{d[f(x,p)]}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p} . \frac{dp}{dx} \quad \text{or} \quad p = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p} . \frac{dp}{dx}$$

Solving the last diff equation, we get the algebraic equation $g(x, p, C) = 0$ or $x = g(p, C)$.

Together with the original equation, they form the following system of equations:

$$\begin{cases} x = g(p, C) \\ y = f(x, p) \end{cases}$$

which is the general solution of the given differential equation in the parametric form. In some cases, when the parameter p can be eliminated from the system, the general solution can be written in the explicit form

$$y = f(x, C)$$

Example 2

Solve the differential equation $2y = 2x^2 + 4xy' + (y')^2$

Solution:

Let $y' = p$, so we can rewrite the equation as

$$2y = 2x^2 + 4xp + (p)^2$$

Differentiate both sides and taking into account that $dy = p\,dx$:

$$2dy = 4xdx + 4pdx + 4xdp + 2pdp$$
$$dy = 2xdx + 2pdx + 2xdp + pdp$$
$$\underline{pdx} = 2xdx + \underline{2pdx} + 2xdp + pdp$$
$$0 = 2xdx + pdx + 2xdp + pdp$$
$$0 = (2x + p)dx + (2x + p)dp$$
$$0 = (2x + p)(dx + dp)$$

- We have two solutions that satisfy the last equation, that is:

$2x + p = 0$

Hence, $2x + y' = 0 \;\Rightarrow y' = -2x, \;\; \Rightarrow dy = -2xdx$

- By integrating this simple equation, we obtain:

$$y_1 = -x^2 + C$$

where C is a constant. To determine the value of C, we substitute this answer in the original differential equation :

$$2(-x^2 + C) = 2x^2 + 4x(-2x) + (-2x)^2$$
$$-2x^2 + 2C = 2x^2 - 8x^2 + 4x^2$$
$$2C = 0 \;\;\Rightarrow\;\; C = 0$$

Thus, the first solution is $y = -x^2$

- Now we consider the second solution: $dx + dp = 0$

Then $\int dx = -\int dp \;\;\Rightarrow\;\; x = -p + C$

- Remember that we have the differential equation : $2y = 2x^2 + 4xp + p^2$

We can substitute the known expression for x (as a function of the parameter p) to find the dependence of y on p:

$$2y = 2(-p + C)^2 + 4(-x + C)p + p^2$$
$$2y = 2p^2 - 4pC + 2C^2 - 3p^2 + 4pC$$
$$2y = 2C^2 - p^2, \Rightarrow y = C^2 - \frac{p^2}{2}$$

- Thus, the second solution is given parametrically by the following system:

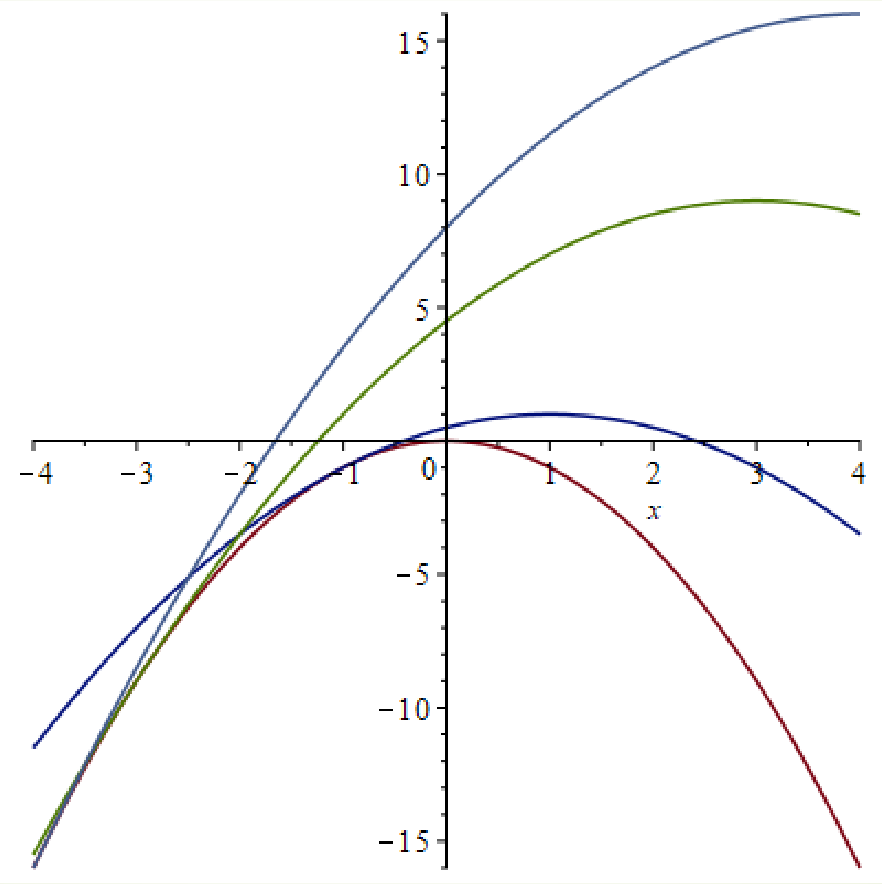
$$\begin{cases} x = -p + C \\ y = C^2 - \frac{p^2}{2} \end{cases}$$

where C is a constant. Eliminating the parameter p, we can write the explicit solution:

$$p = C - x \quad \Rightarrow \quad y_2 = C^2 - \frac{(C - x)^2}{2}$$

- The final answer is given by $y = -x^2$, $y = C^2 - \frac{(C-x)^2}{2}$

- Figure of the solution
- y_1 is the envelope of y_2



Exercise 1

Solve the differential equation :

1. $y = xy' + (y')^2$
2. $y = x^2p^4 + 2xp$

Type 3: Implicit Diff Equation of Type $\boldsymbol{x = f(y, y')}$

The variable x is expressed explicitly in terms of y and the derivative y'.

- Let the parameter $p = y' = \frac{dy}{dx}$.
- Differentiate the equation $x = f(y, y') = f(y, p)$ with respect to y.

This produces: $\frac{dx}{dy} = \frac{d[f(y,p)]}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p} \cdot \frac{dp}{dy}$

As $\frac{dx}{dy} = \frac{1}{p}$, the last expression can be written as follows: $\frac{1}{p} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p} \cdot \frac{dp}{dy}$

- We obtain an explicit differential equation such that its general solution is given by the function $g(y, p, C) = 0$ or $y = g(p, C)$ where C is a constant.
- Thus, the general solution of the original implicit differential equation is defined in the parametric form by the system of two algebraic equations:

$$\begin{cases} y = g(p, C) \\ x = f(y, p) \end{cases}$$

- If the parameter p can be eliminated from the system, the general solution is given in the explicit form $x = f(y, C)$

Type 4: Implicit Diff Equation of Type $\boldsymbol{y = f(y')}$

- The equation of this kind does not contain the variable x and can be solved the similar way. Using the parameter $p = y' = \frac{dy}{dx}$, we can write $dx = \frac{1}{p}dy$.
- Then it follows from the equation that $dx = \frac{1}{p}\frac{df}{dp}dp$
- Integrating the last expression gives the general solution of the original implicit equation in parametric form:

$$\begin{cases} x = \int \frac{1}{p}\frac{df}{dp}dp + C \\ y = f(p) \end{cases}$$

Example 3:

Find the general solution of the differential equation $y = \ln[25 + (y')^2]$

Solution.

Using the parameter p we rewrite this equation : $y = \ln[25 + p^2]$

Take the differentials of both sides: $dy = \frac{2pdp}{25+p^2}$

As $dy = pdx$, we get

$$pdx = \frac{2pdp}{25 + p^2}$$

$$dx = \frac{2dp}{25 + p^2}$$

$$x = 2\int \frac{dp}{25 + p^2}$$

$$x = \frac{2}{5}\arctan\frac{p}{5} + C$$

So we have the following parametric representation of the solution of the differential equation:

$$\begin{cases} x = \frac{2}{5}\arctan\frac{p}{5} + C \\ y = \ln(25 + p^2) \end{cases}$$

where C is an arbitrary constant.

Type 5: Implicit Diff Equation of Type $\boldsymbol{x = f(y')}$

- Here the differential equation does not contain the variable y.
- Using the parameter $p = y' = \frac{dy}{dx}$,

it's easy to construct the general solution of the equation.

As $dx = d[f(p)] = \frac{df}{dp}dp$ and $dy = p\,dx$

- then the following relationship holds:

$$dy = p\frac{df}{dp}dp$$

- Integrating the last equation gives the general solution in the parametric form:

$$\begin{cases} y = \int p\frac{df}{dp}dp + C \\ x = f(p) \end{cases}$$

Example 4

Find the general solution of the equation $9(y')^2 - 4x = 0$

Solution.

Let the parameter $p = y'$ and write the equation in the form: $x = \frac{9}{4}p^2$

By taking differentials of both sides, we obtain:

$$dx = \frac{9}{4}2pdp = \frac{9}{2}pdp$$

Since $dy = pdx$, the last expression can be presented as

$$\frac{dy}{p} = \frac{9}{2}p\,dp \;\Rightarrow\; dy = \frac{9}{2}p^2\,dp$$

By integrating we find the dependence of the variable y on the parameter p :

$y = \int \frac{9}{2}p^2\,dp = \frac{3}{2}p^3 + C$, where C is a constant.

- Thus, we get the general solution of the equation in parametric form:

$$\begin{cases} y = \frac{3}{2}p^3 + C \\ x = \frac{9}{2}p^2 \end{cases}$$

- We can eliminate the parameter p from this system. It follows from the second equation that

$$p^2 = \frac{4}{9}x, \qquad \Rightarrow \quad p = \pm\frac{2}{3}x^{\frac{1}{2}}$$

- Substituting this in the first equation, we obtain the general solution as the explicit function $y = f(x)$:

$$y = \frac{3}{2}\left(\pm\frac{2}{3}x^{\frac{1}{2}}\right)^3 + C = \pm\frac{4}{9}x^{\frac{3}{2}} + C$$