FUNGSI GAMMA DAN BETA

Fs Khusus

MATEMATIKA REKAYASA 1

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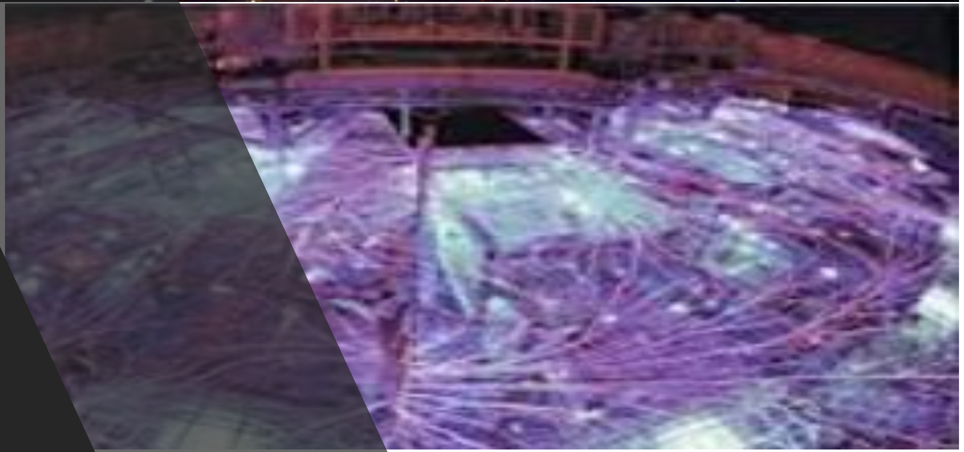


1. Definisi Fs Gamma

Fungsi Gamma

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

dan

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty e^{-t} t^{x} dt \\
&= \lim_{B\to\infty} \int_0^B e^{-t} t^{x} dt \\
&= \lim_{B\to\infty} \left[-e^{-t} t^{x} \Big|_0^B + x \int_0^B e^{-t} t^{x-1} dt \right] \\
&= x \lim_{B\to\infty} \int_0^B e^{-t} t^{x-1} dt \\
&= x\Gamma(x)
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\Gamma(2) &= 1 \cdot \Gamma(1) \\
&= \int_0^\infty e^{-t} dt = 1 \\
\Gamma(3) &= 2 \cdot \Gamma(2) = 2 \cdot 1 \\
\Gamma(4) &= 3 \cdot \Gamma(3) = 3 \cdot 2 \cdot 1 \\
&\vdots \\
\Gamma(x) &= (x-1)!
\end{aligned}$$

Konvergensi Fs Gamma

Untuk setiap x > 0

$$\int_0^\infty e^{-t}t^{x-1}dt$$ adalah konvergen

Bukti

$$\int_0^\infty e^{-xt}dt$$

$$\begin{aligned}
\int_0^\infty e^{-xt}dt &= \lim_{B\to\infty}\int_0^B e^{-xt}dt \\
&= \lim_{B\to\infty}\left[\frac{-e^{xt}}{x}\right]\Big|_0^B \\
&= \lim_{B\to\infty}\frac{-e^{-Bx}-1}{x} \\
&= \frac{1}{x}\left[\lim_{B\to\infty}-e^{-Bx}\to 0 - \lim_{B\to\infty}1\to 1\right] \\
&= \frac{1}{x}
\end{aligned}$$

Untuk n bilangan natural

$$\lim_{t\to\infty} \frac{t^{n-1}}{e^{\frac{1}{2}t}} = 0$$

Bukti

$$\lim_{t\to\infty} \frac{t^{n-1}}{e^{\frac{1}{2}t}} = \lim_{t\to\infty} \frac{(n-1)t^{n-2}}{\frac{1}{2}e^{\frac{1}{2}t}}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} t^{n-1} = 0$$

$$\lim_{t\to\infty} \frac{t^{n-1}}{e^{\frac{1}{2}t}} = \lim_{t\to\infty} \frac{0}{(\frac{1}{2})^n e^{\frac{1}{2}t}} = 0$$

Turunan Fs Gamma

$$\Gamma'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty e^{-t}t^{x-1}dt = \int_0^\infty e^{-t}t^{x-1}\ln(t)dt$$

$$\Gamma''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\Gamma'(x) = \int_0^\infty e^{-t}t^{x-1}(\ln(t))^2dt$$

Sifat - pengembangan Fs Gamma

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Rightarrow \Gamma(x) = \frac{\Gamma(x+1)}{x}$$

$$\Rightarrow \Gamma(0) = \frac{\Gamma(1)}{0}$$

$$\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}} = \frac{\Gamma(\frac{1}{2})}{-\frac{3}{2}\cdot-\frac{1}{2}} = \frac{4}{3}\sqrt{\pi}$$

$$\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma(-\frac{3}{2})}{-\frac{5}{2}} = \frac{\Gamma(-\frac{1}{2})}{-\frac{5}{2}\cdot-\frac{3}{2}} = \frac{\Gamma(\frac{1}{2})}{-\frac{5}{2}\cdot-\frac{3}{2}\cdot-\frac{1}{2}} = -\frac{8}{15}\sqrt{\pi}$$

$$\Gamma(-1) = \frac{\Gamma(0)}{-1} = \frac{\Gamma(1)}{-1\cdot 0}$$

$$\Gamma(-2) = \frac{\Gamma(-1)}{-2} = \frac{\Gamma(0)}{-1\cdot-2} = \frac{\Gamma(1)}{2\cdot 0}$$

Sifat - pengembangan Fs Gamma

$$\Gamma(z+n) = (z+n-1)\Gamma(z+n-1) = (z+n-1)(z+n-2)\ldots(z+1)(z)\Gamma(z)$$

$$\Gamma(z) = \frac{\Gamma(z+n)}{P_n(z)}$$

Pn(x) : Faktorial Pochhammer

$$x^{(n)} = x(x+1)(x+2)\ldots(x+n-1)$$

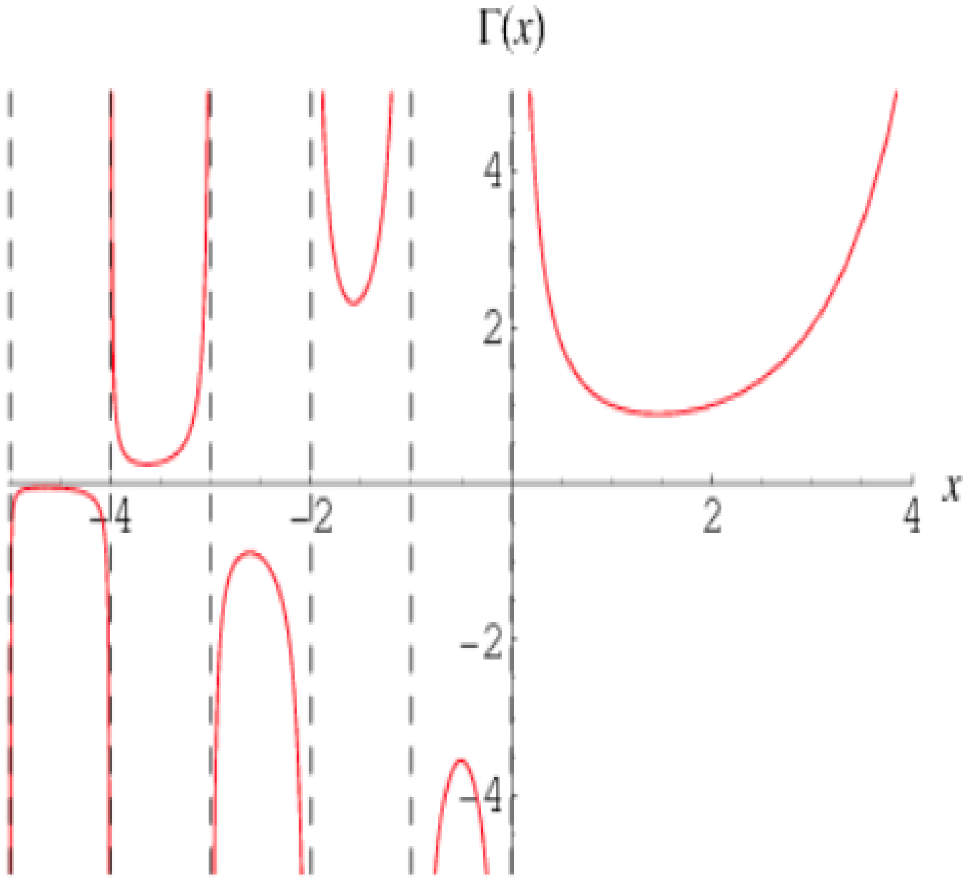


TABLE 15.1

N	$\Gamma(N)$
1.00	1.0000
1.10	0.9514
1.20	0.9182
1.30	0.8975
1.40	0.8873
1.50	0.8862
1.60	0.8935
1.70	0.9086
1.80	0.9314
1.90	0.9618
2.00	1.0000

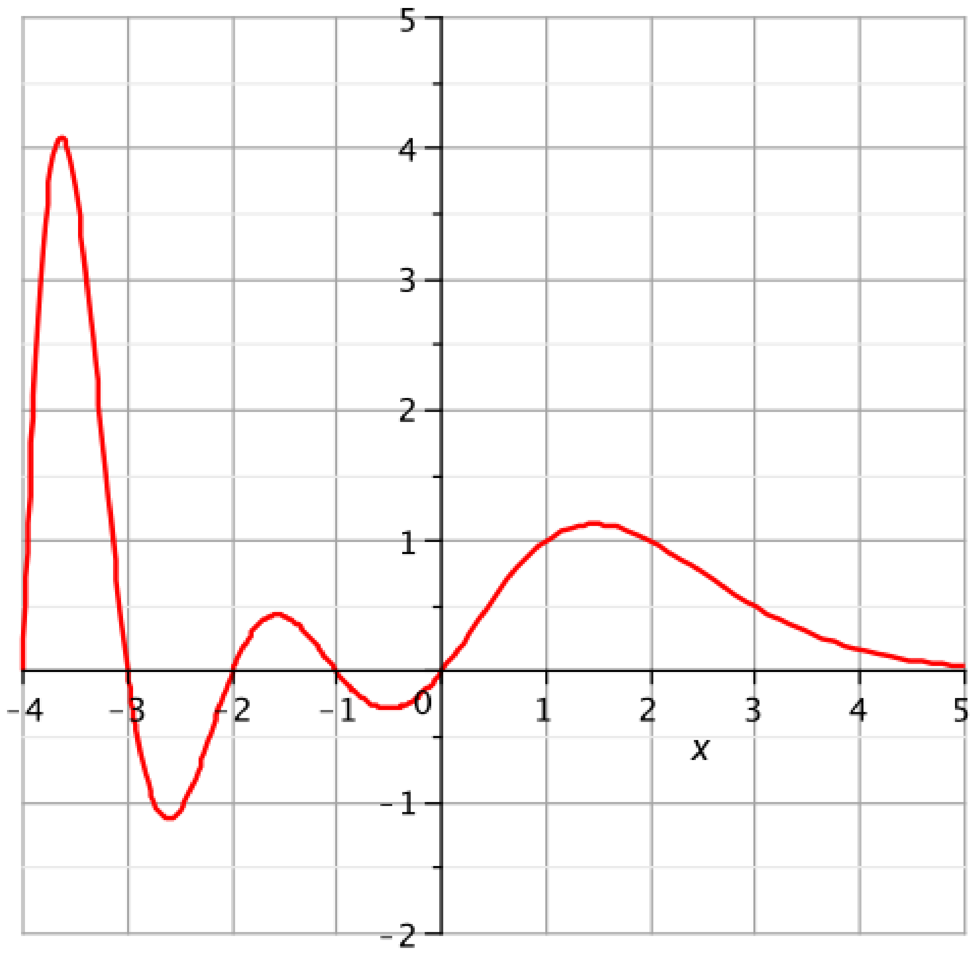
Fs Gamma yg tidak lengkap

$$\gamma(x,\alpha) = \int_0^\alpha e^{-t} t^{x-1} dt$$

$$\Gamma(x,\alpha) = \int_\alpha^\infty e^{-t} t^{x-1} dt$$

$$\gamma(x,\alpha) + \Gamma(x,\alpha) = \Gamma(x)$$

$1/\Gamma(x)$



2. Definisi Fs Beta

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

Hubungan Fs Gamma dan Beta

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Contoh $\int_1^3 (x-1)^{10}(x-3)^3 dx.$

Dengan menggunakan $t = \frac{x-1}{2}$

$$\Rightarrow x = 2t+1 \Rightarrow dx = 2t$$

diperoleh

$$\int_0^1 (2t)^{10}(2t-2)^3 2dt = -2^{14}\int_0^1 t^{10}(1-t)^3 dt =$$

$$= -2^{14}\beta(11,4) = -2^{14}\frac{\Gamma(11)\Gamma(4)}{\Gamma(15)} = -2^{14}\cdot\frac{10!\cdot 3!}{14!}$$

Fs Beta sbg pengembangan Faktorial

$$\binom{n}{k} = \frac{1}{(n+1)\beta(n-k+1, k+1)}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n+1)\cdot n!}{(n+1)\cdot k!\cdot(n-k)!} = \frac{(n+1)!}{(n+1)\cdot k!(n-k)} =$$

$$= \frac{1}{n+1}\frac{\Gamma(n+2)}{\Gamma(k+1)\Gamma(n-k+1)} = \frac{1}{n+1}\frac{1}{\beta(n-k+1, k+1)}$$

$$\binom{3}{2} = \frac{1}{4\beta(2,3)} = \frac{\Gamma(5)}{4\Gamma(2)\Gamma(3)} = 3$$

Rumus Stirling

$$n! \approx n^n e^{-n}\sqrt{2\pi n}$$

$$\lim_{n\to\infty} \frac{n}{n^n e^{-n}\sqrt{2\pi n}} = 1$$

$$\Gamma(n+1) = \int_0^\infty e^{-t} t^n dt = n!$$

. $t = nk,$

$$= \int_0^\infty e^{-nk}(nk)^n n dk$$

$$= (n)^{n+1} = \int_0^\infty e^{-nk} k^n dk$$

Subs. s dg $(k-1)\sqrt{n},$

$$= n^{n+1}\int_{-\sqrt{n}}^\infty e^{-n\left(\frac{s}{\sqrt{n}}+1\right)}\left(\frac{s}{\sqrt{n}}+1\right)^n \frac{1}{\sqrt{n}} ds$$

$$= n^n\sqrt{n}e^{-n}\int_{-\sqrt{n}}^\infty e^{-s\sqrt{n}} e^{n\log\left(1+\frac{s}{\sqrt{n}}\right)} ds$$

Deret Taylor dr $\log(1+r),$

$$\log(1+r) = r - \frac{r^2}{2} + \frac{r^3}{3} - \frac{r^4}{4} + \ldots$$

Fs Error

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{1!3} + \frac{x^5}{2!5} - \frac{x^7}{3!7} + - \cdots \right)$$

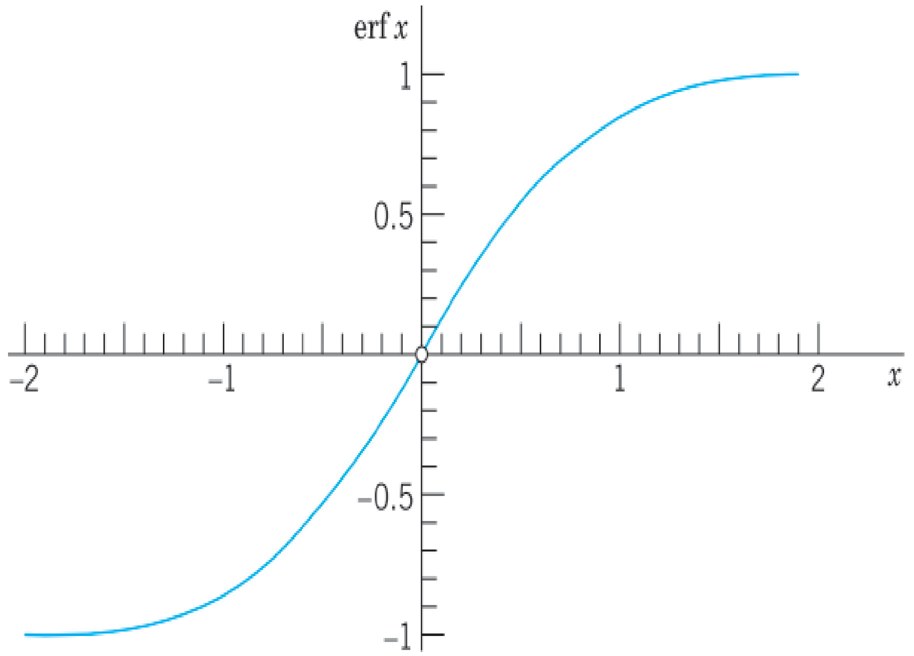


Fig. 554. Error function

Complement Fs Error

$$\operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

Beberapa notasi dalam diff

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

$$\frac{\partial^2 f}{\partial x\ \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y\ \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}.$$

$$\frac{\partial^2 z}{\partial x\ \partial y} = \frac{\partial^2 z}{\partial y\ \partial x}\ .$$

Contoh dan Latihan Soal

Evaluate each of the following:

(a) $$\frac{\Gamma(6)}{2\Gamma(3)} = \frac{5!}{2\cdot 2!} = \frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 2} = 30$$

(b) $$\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{3}{4}$$

(c) $$\frac{\Gamma(3)\Gamma(2.5)}{\Gamma(5.5)} = \frac{2!(1.5)(0.5)\Gamma(0.5)}{(4.5)(3.5)(2.5)(1.5)(0.5)\Gamma(0.5)} = \frac{16}{315}$$

(d) $$\frac{6\Gamma\left(\frac{8}{3}\right)}{5\Gamma\left(\frac{2}{3}\right)} = \frac{6\left(\frac{5}{3}\right)\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}{5\Gamma\left(\frac{2}{3}\right)} = \frac{4}{3}$$

Evaluate each integral.

(a) $$\int_0^\infty x^3 e^{-x}\,dx = \Gamma(4) = 3! = 6$$

(b) $$\int_0^\infty x^6 e^{-2x}\,dx$$

Catat semua Informasi tambahan dari perkuliahan – online

Terimakasih