The Solution of Non-Linier Equation

Matematika Rekayasa 1

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Ref : Erwin Kreyszig-Advanced Engineering Mathematics
http://numericalmethods.eng.usf.edu

CapaianPembelajaran:

- Mampu menyelesaikan akar akar persamaan non linier dengan menggunakan metode regula falsi, biseksi, newton raphson dan secant
- Mampu membandingkan kelebihan dan kekurangan keempat metode tersebut

Solution of equation by iteration

Error in Numerical Result

Formulas for Errors. If $\tilde{a}$ is an approximate value of a quantity whose exact value is a, we call the difference

$$\epsilon = a - \tilde{a} \tag{4}$$

the **error** of $\tilde{a}$. Hence

$$a = \tilde{a} + \epsilon, \qquad \text{True value} = \text{Approximation} + \text{Error}. \tag{4*}$$

For instance, if $\tilde{a} = 10.5$ is an approximation of $a = 10.2$, its error is $\epsilon = -0.3$. The error of an approximation $\tilde{a} = 1.60$ of $a = 1.82$ is $\epsilon = 0.22$.

The **relative error** ϵ_r of $\tilde{a}$ is defined by

$$\epsilon_r = \frac{\epsilon}{a} = \frac{a - \tilde{a}}{a} = \frac{\text{Error}}{\text{True value}} \qquad (a \neq 0). \tag{5}$$

This looks useless because a is unknown. But if $|\epsilon|$ is much less than $|\tilde{a}|$, then we can use $\tilde{a}$ instead of a and get

$$\epsilon_r \approx \frac{\epsilon}{\tilde{a}}\,. \tag{5'}$$

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Prinsip dasar error

Didalam setiap metode numerik harus terdapat perhitungan error. Jika tidak terdapat formulasi eror, maka hasil numerik tersebut akan menjadi complicated

The False-Position Method (Regula-Falsi)

Basis of False Position Method

We can approximate the solution by doing a *linear interpolation* between $f(x_u)$ and $f(x_l)$

Find x_r such that $l(x_r)=0$, where $l(x)$ is the linear approximation of $f(x)$ between x_l and x_u

Derive x_r using similar triangles

$$x_r = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$$

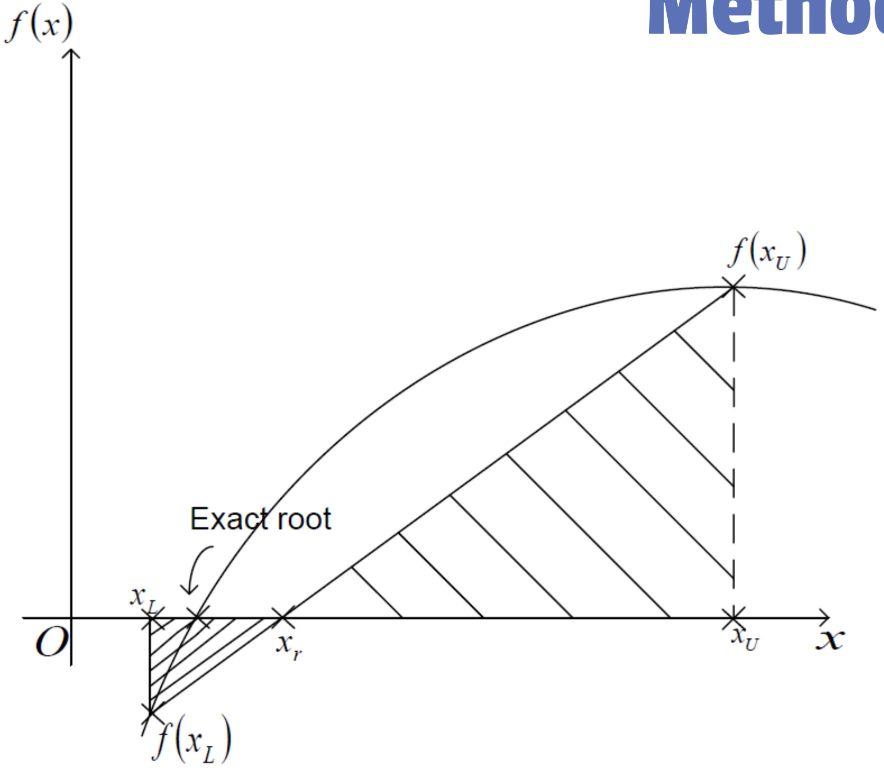


Figure 1 False-Position Method

Based on two similar triangles, shown in Figure 1, one gets

$$\frac{0 - f(x_L)}{x_r - x_L} = \frac{0 - f(x_U)}{x_r - x_U}$$

From Equation (4), one obtains

$$(x_r - x_L)f(x_U) = (x_r - x_U)f(x_L)$$

$$x_U f(x_L) - x_L f(x_U) = x_r\{f(x_L) - f(x_U)\}$$

The above equation can be solved to obtain the next predicted root x_m as

$$x_r = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$$

The above equation, through simple algebraic manipulations, can also be expressed as

$$x_r = x_U - \frac{f(x_U)}{\left\{\frac{f(x_L) - f(x_U)}{x_L - x_U}\right\}} \longrightarrow x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

or

$$x_r = x_L - \frac{f(x_L)}{\left\{\frac{f(x_U) - f(x_L)}{x_U - x_L}\right\}}$$

Step 1

1. Choose x_L and x_U as two guesses for the root such that $f(x_L)f(x_U) < 0$, or in other words, $f(x)$ changes sign between x_L and x_U.

Step 2

2. Estimate the root, x_r of the equation $f(x) = 0$ as

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

Step 3

3. Now check the following

If $f(x_L)f(x_r) < 0$, then the root lies between x_L and x_r; then $x_L = x_L$ and $x_U = x_r$.

If $f(x_L)f(x_r) > 0$, then the root lies between x_r and x_U; then $x_L = x_r$ and $x_U = x_U$.

If $f(x_L)f(x_r) = 0$, then the root is x_r. Stop the algorithm.

Step 4

4. Find the new estimate of the root

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

Find the absolute relative approximate error as

$$|\epsilon_a| = \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right| \times 100$$

where

x_r^{new} = estimated root from present iteration

x_r^{old} = estimated root from previous iteration

Step 5

5. Compare the absolute relative approximate error $|\in_a|$ with the pre-specified relative error tolerance $\in_s$. If $|\in_a| > \in_s$, then go to step 3, else stop the algorithm. Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Note that the false-position and bisection algorithms are quite similar. The only difference is the formula used to calculate the new estimate of the root x_r as shown in steps #2 and #4!

Example-1

Find a root of an equation $f(x) = x^3 - x - 1$ using False Position method

Solution:

Here $x^3 - x - 1 = 0$

Let $f(x) = x^3 - x - 1$

Here

x	0	1	2
$f(x)$	-1	-1	5

1^{st} iteration :

Here $f(1) = -1 < 0$ and $f(2) = 5 > 0$

$\therefore$ Now, Root lies between $x_0 = 1$ and $x_1 = 2$

$$x_2 = x_0 - f(x_0) \cdot \frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

$$x_2 = 1 - (-1) \cdot \frac{2 - 1}{5 - (-1)}$$

$$x_2 = 1.16667$$

$$f(x_2) = f(1.16667) = -0.5787 < 0$$

2^{nd} iteration :

Here $f(1.16667) = -0.5787 < 0$ and $f(2) = 5 > 0$

$\therefore$ Now, Root lies between $x_0 = 1.16667$ and $x_1 = 2$

$$x_3 = x_0 - f(x_0) \cdot \frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

$$x_3 = 1.16667 - (-0.5787) \cdot \frac{2 - 1.16667}{5 - (-0.5787)}$$

$$x_3 = 1.25311$$

$$f(x_3) = f(1.25311) = -0.28536 < 0$$

3^{rd} iteration :

Here $f(1.25311) = -0.28536 < 0$ and $f(2) = 5 > 0$

$\therefore$ Now, Root lies between $x_0 = 1.25311$ and $x_1 = 2$

$$x_4 = x_0 - f(x_0) \cdot \frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

$$x_4 = 1.25311 - (-0.28536) \cdot \frac{2 - 1.25311}{5 - (-0.28536)}$$

$$x_4 = 1.29344$$

$$f(x_4) = f(1.29344) = -0.12954 < 0$$

4^{th} iteration :

Here $f(1.29344) = -0.12954 < 0$ and $f(2) = 5 > 0$

$\therefore$ Now, Root lies between $x_0 = 1.29344$ and $x_1 = 2$

$$x_5 = x_0 - f(x_0) \cdot \frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

$$x_5 = 1.29344 - (-0.12954) \cdot \frac{2 - 1.29344}{5 - (-0.12954)}$$

$$x_5 = 1.31128$$

$$f(x_5) = f(1.31128) = -0.05659 < 0$$

9^{th} iteration :

Here $f(1.32428) = -0.00187 < 0$ and $f(2) = 5 > 0$

$\therefore$ Now, Root lies between $x_0 = 1.32428$ and $x_1 = 2$

$$x_{10} = x_0 - f(x_0) \cdot \frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

$$x_{10} = 1.32428 - (-0.00187) \cdot \frac{2 - 1.32428}{5 - (-0.00187)}$$

$$x_{10} = 1.32453$$

$$f(x_{10}) = f(1.32453) = -0.00079 < 0$$

10^{th} iteration :

Here $f(1.32453) = -0.00079 < 0$ and $f(2) = 5 > 0$

$\therefore$ Now, Root lies between $x_0 = 1.32453$ and $x_1 = 2$

$$x_{11} = x_0 - f(x_0) \cdot \frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

$$x_{11} = 1.32453 - (-0.00079) \cdot \frac{2 - 1.32453}{5 - (-0.00079)}$$

$$x_{11} = 1.32464$$

$$f(x_{11}) = f(1.32464) = -0.00034 < 0$$

Approximate root of the equation $x^3 - x - 1 = 0$ using False Position mehtod is 1.32464

n	x_0	$f(x_0)$	x_1	$f(x_1)$	x_2	$f(x_2)$
1	1	-1	2	5	1.16667	-0.5787
2	1.16667	-0.5787	2	5	1.25311	-0.28536
3	1.25311	-0.28536	2	5	1.29344	-0.12954
4	1.29344	-0.12954	2	5	1.31128	-0.05659
5	1.31128	-0.05659	2	5	1.31899	-0.0243
6	1.31899	-0.0243	2	5	1.32228	-0.01036
7	1.32228	-0.01036	2	5	1.32368	-0.0044
8	1.32368	-0.0044	2	5	1.32428	-0.00187
9	1.32428	-0.00187	2	5	1.32453	-0.00079
10	1.32453	-0.00079	2	5	1.32464	-0.00034

Bisection Method

Basis of Bisection Method

Theorem An equation f(x)=0, where f(x) is a real continuous function, has at least one root between x_l and x_u if $f(x_l)\,f(x_u) < 0$.

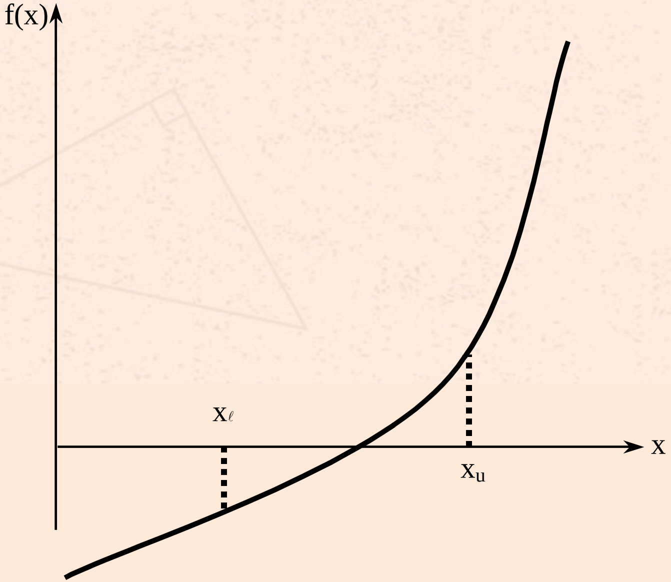


Figure 1 At least one root exists between the two points if the function is real, continuous, and changes sign.

Basis of Bisection Method

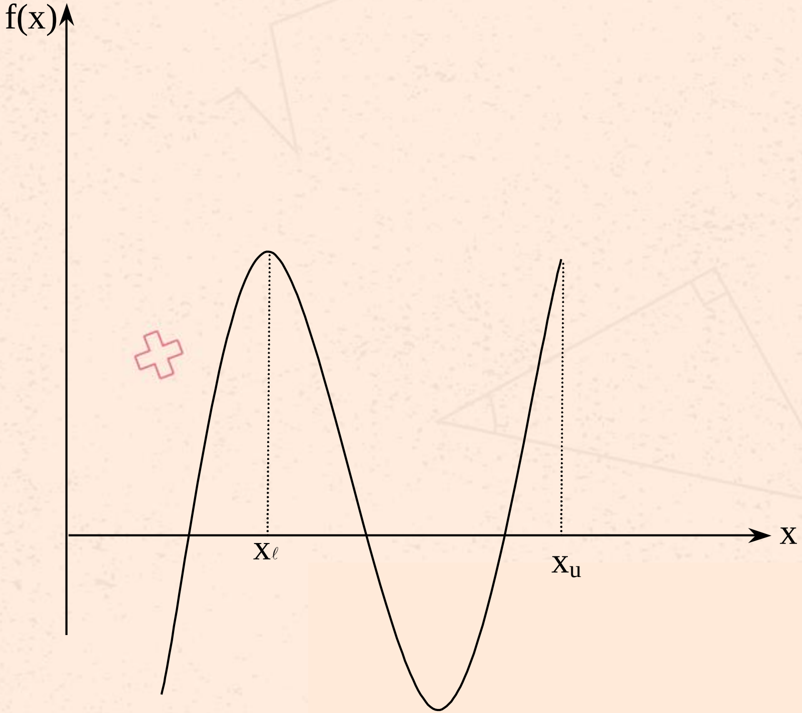
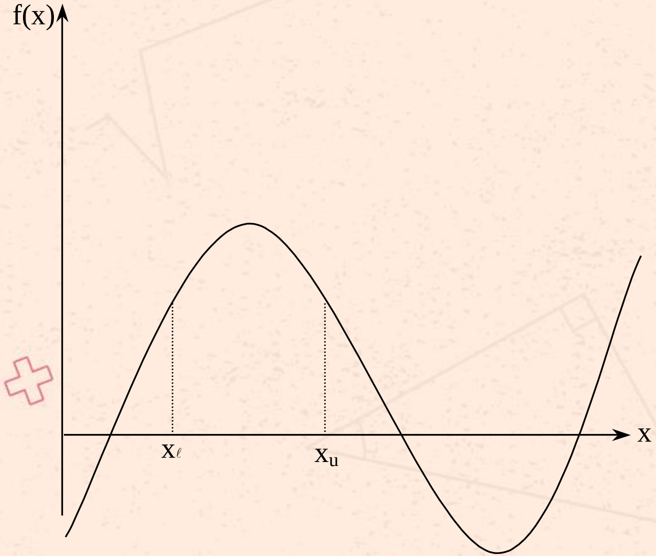


Figure 2 If function $f(x)$ does not change sign between two points, roots of the equation $f(x)=0$ may still exist between the two points.

Basis of Bisection Method



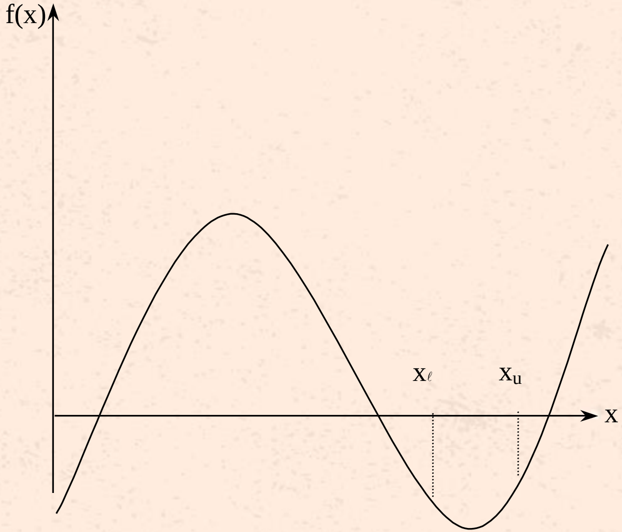


Figure 3 If the function $f(x)$ does not change sign between two points, there may not be any roots for the equation $f(x)=0$ between the two points.

Basis of Bisection Method

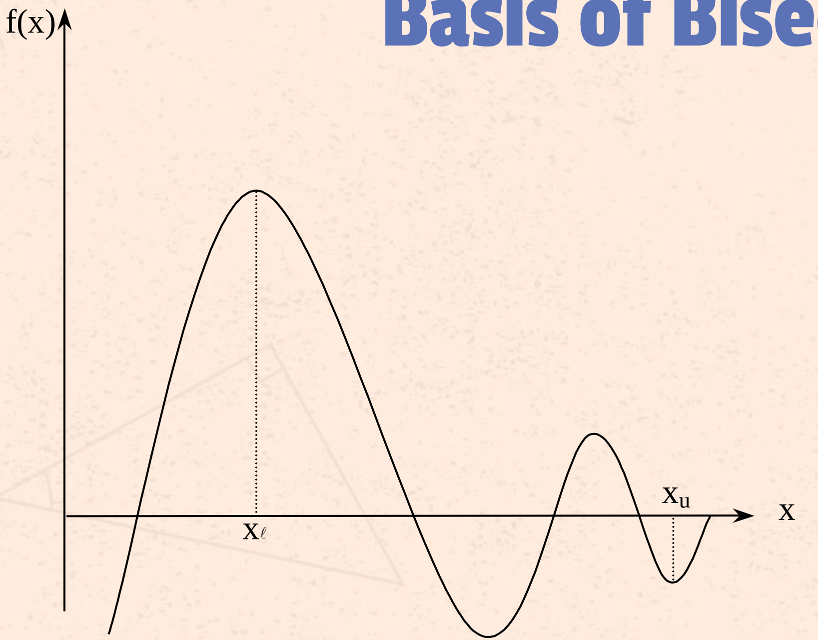


Figure 4 If the function $f(x)$ changes sign between two points, more than one root for the equation $f(x)=0$ may exist between the two points.

Step 1

Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)\ f(x_u) < 0$, or in other words, f(x) changes sign between x_ℓ and x_u. This was demonstrated in Figure 1.

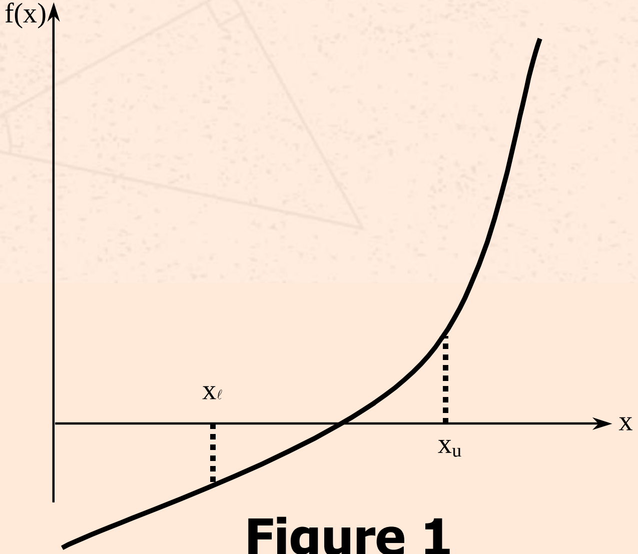


Figure 1

Step 2

Estimate the root, x_m of the equation f (x) = 0 as the mid point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$

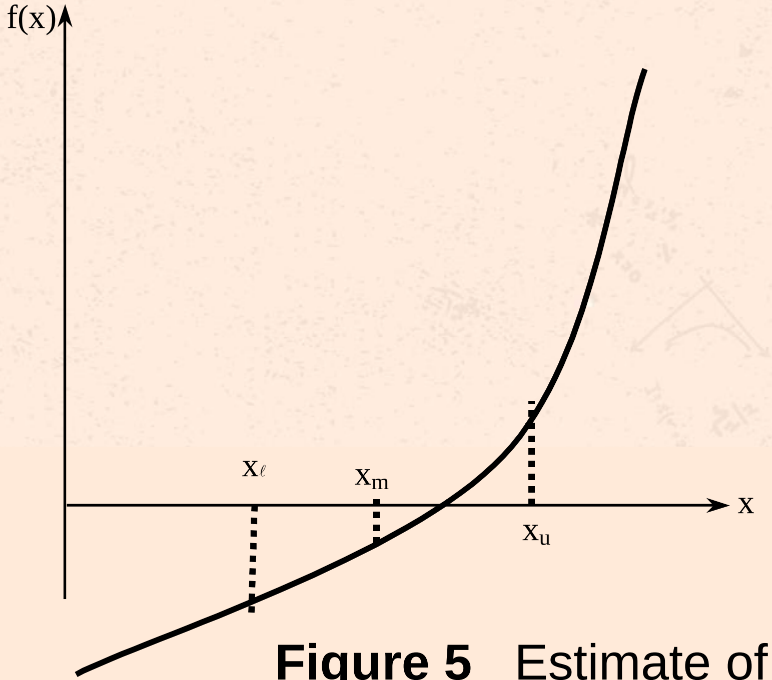


Figure 5 Estimate of x_m

Step 3

Now check the following

a) If $f(x_l)f(x_m) < 0$, then the root lies between x_ℓ and x_m; then $x_\ell = x_\ell$; $x_u = x_m$.

b) If $f(x_l)f(x_m) > 0$, then the root lies between x_m and x_u; then $x_\ell = x_m$; $x_u = x_u$.

c) If $f(x_l)f(x_m) = 0$; then the root is x_m. Stop the algorithm if this is true.

Step 4

Find the new estimate of the root

$$x_m = \frac{x_\ell + x_u}{2}$$

Find the absolute relative approximate error

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

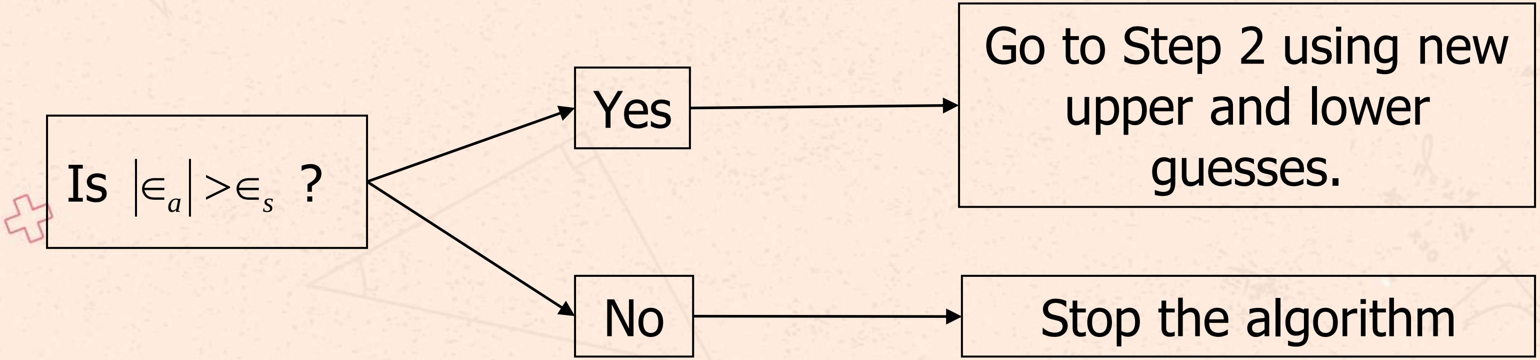
where

$$x_m^{old} = \text{previous estimate of root}$$

$$x_m^{new} = \text{current estimate of root}$$

Step 5

Compare the absolute relative approximate error $|\in_a|$ with the pre-specified error tolerance $\in_s$.



Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Example-1

1. Find a root of an equation $f(x) = x^3 - x - 1$ using Bisection method

Solution:

Here $x^3 - x - 1 = 0$

Let $f(x) = x^3 - x - 1$

Here

x	0	1	2
$f(x)$	-1	-1	5

1^{st} iteration :

Here $f(1) = -1 < 0$ and $f(2) = 5 > 0$

$\therefore$ Now, Root lies between 1 and 2

$$x_0 = \frac{1+2}{2} = 1.5$$

$$f(x_0) = f(1.5) = 0.875 > 0$$

2^{nd} iteration :

Here $f(1) = -1 < 0$ and $f(1.5) = 0.875 > 0$

$\therefore$ Now, Root lies between 1 and 1.5

$$x_1 = \frac{1+1.5}{2} = 1.25$$

$$f(x_1) = f(1.25) = -0.29688 < 0$$

3^{rd} iteration :

Here $f(1.25) = -0.29688 < 0$ and $f(1.5) = 0.875 > 0$

$\therefore$ Now, Root lies between 1.25 and 1.5

$$x_2 = \frac{1.25 + 1.5}{2} = 1.375$$

$$f(x_2) = f(1.375) = 0.22461 > 0$$

4^{th} iteration :

Here $f(1.25) = -0.29688 < 0$ and $f(1.375) = 0.22461 > 0$

$\therefore$ Now, Root lies between 1.25 and 1.375

$$x_3 = \frac{1.25 + 1.375}{2} = 1.3125$$

$$f(x_3) = f(1.3125) = -0.05151 < 0$$

6^{th} iteration :

Here $f(1.3125) = -0.05151 < 0$ and $f(1.34375) = 0.08261 > 0$

$\therefore$ Now, Root lies between 1.3125 and 1.34375

$$x_5 = \frac{1.3125 + 1.34375}{2} = 1.32812$$

$$f(x_5) = f(1.32812) = 0.01458 > 0$$

7^{th} iteration :

Here $f(1.3125) = -0.05151 < 0$ and $f(1.32812) = 0.01458 > 0$

$\therefore$ Now, Root lies between 1.3125 and 1.32812

$$x_6 = \frac{1.3125 + 1.32812}{2} = 1.32031$$

$$f(x_6) = f(1.32031) = -0.01871 < 0$$

8^{th} iteration :

Here $f(1.32031) = -0.01871 < 0$ and $f(1.32812) = 0.01458 > 0$

$\therefore$ Now, Root lies between 1.32031 and 1.32812

$$x_7 = \frac{1.32031 + 1.32812}{2} = 1.32422$$

$$f(x_7) = f(1.32422) = -0.00213 < 0$$

9^{th} iteration :

Here $f(1.32422) = -0.00213 < 0$ and $f(1.32812) = 0.01458 > 0$

$\therefore$ Now, Root lies between 1.32422 and 1.32812

$$x_8 = \frac{1.32422 + 1.32812}{2} = 1.32617$$

$$f(x_8) = f(1.32617) = 0.00621 > 0$$

10^{th} iteration :

Here $f(1.32422) = -0.00213 < 0$ and $f(1.32617) = 0.00621 > 0$

$\therefore$ Now, Root lies between 1.32422 and 1.32617

$$x_9 = \frac{1.32422 + 1.32617}{2} = 1.3252$$

$$f(x_9) = f(1.3252) = 0.00204 > 0$$

11^{th} iteration :

Here $f(1.32422) = -0.00213 < 0$ and $f(1.3252) = 0.00204 > 0$

$\therefore$ Now, Root lies between 1.32422 and 1.3252

$$x_{10} = \frac{1.32422 + 1.3252}{2} = 1.32471$$

$$f(x_{10}) = f(1.32471) = -0.00005 < 0$$

Approximate root of the equation $x^3 - x - 1 = 0$ using Bisection mehtod is 1.32471

n	a	$f(a)$	b	$f(b)$	$c = \frac{a+b}{2}$	$f(c)$
1	1	-1	2	5	1.5	0.875
2	1	-1	1.5	0.875	1.25	-0.29688
3	1.25	-0.29688	1.5	0.875	1.375	0.22461
4	1.25	-0.29688	1.375	0.22461	1.3125	-0.05151
5	1.3125	-0.05151	1.375	0.22461	1.34375	0.08261
6	1.3125	-0.05151	1.34375	0.08261	1.32812	0.01458
7	1.3125	-0.05151	1.32812	0.01458	1.32031	-0.01871
8	1.32031	-0.01871	1.32812	0.01458	1.32422	-0.00213
9	1.32422	-0.00213	1.32812	0.01458	1.32617	0.00621
10	1.32422	-0.00213	1.32617	0.00621	1.3252	0.00204
11	1.32422	-0.00213	1.3252	0.00204	1.32471	-0.00005

Tugas Penyelesaian Persamaan Non Linier

Cari akar akar persamaan non linier dibawah dengan menggunakan metode regula falsi dan biseksi

1. $2x^3-2x-5 = 0$, interval [1,2]
2. $f(x) = 3x + \sin x - e^x = 0$

   $x_0 = 0, x_1 = 1, f_0 \times f_1 < 0$

3. $5\sin^2 x - 8\cos^5 x = 0$, interval [0.5, 1.5]
4. $(x-2)^2 - \ln x = 0$, interval [1,2]