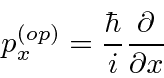
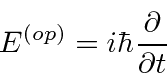
## Operators

The Schrödinger equation comes directly out of our understanding of wave packets. To get from wave packets to a differential equation, we use the new concept of (linear) [operators.](http://quantummechanics.ucsd.edu/ph130a/130_notes/node100.html#section:operators) We determine the momentum and energy operators by requiring that, when an operator for some variable \bgroup\color{black}$v$\egroupacts on our simple wavefunction, we get \bgroup\color{black}$v$\egrouptimes the same wave function.



\begin{displaymath}\bgroup\color{black}p_x^{(op)}e^{i(\vec{p}\cdot\vec{x}-Et)/\h...
...ec{x}-Et)/\hbar}
=p_xe^{i(\vec{p}\cdot\vec{x}-Et)/\hbar}\egroup\end{displaymath}



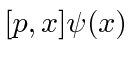
\begin{displaymath}\bgroup\color{black}E^{(op)}e^{i(\vec{p}\cdot\vec{x}-Et)/\hba...
...\vec{x}-Et)/\hbar}
=Ee^{i(\vec{p}\cdot\vec{x}-Et)/\hbar}\egroup\end{displaymath}

## Commutators

Operators (or variables in quantum mechanics) do not necessarily commute. We can see our first example of that now that we have a few operators. We **define the commutator** to be

|  |
| --- |
| \bgroup\color{black}$\displaystyle [p,x]\equiv px-xp$\egroup |

(using \bgroup\color{black}$p$\egroupand \bgroup\color{black}$x$\egroupas examples.)

We will now **compute the commutator** between \bgroup\color{black}$p$\egroupand \bgroup\color{black}$x$\egroup. Because \bgroup\color{black}$p$\egroupis represented by a differential operator, we must do this carefully. Lets think of the commutator as a (differential) operator too, as generally it will be. To make sure that we keep all the \bgroup\color{black}${\partial\over\partial x}$\egroupthat we need, we will compute then remove the at the end to see only the commutator.

\begin{eqnarray*}[p,x]\psi(x)&=&px\psi(x)-xp\psi(x)
={\hbar\over i}{\partial\ove...
...\partial\psi(x)\over\partial x}\right)
={\hbar\over i}\psi(x)\\
\end{eqnarray*}

So, removing the we used for computational purposes, we get the commutator.

|  |
| --- |
| \bgroup\color{black}$\displaystyle [p,x]={\hbar\over i}$\egroup |

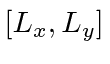
Later we will learn to derive the uncertainty relation for two variables from their commutator. Physical variable with zero commutator have no uncertainty principle and we can know both of them at the same time.

We will also use commutators to solve several important problems.

We can compute the **same commutator in momentum space**.

\begin{eqnarray*}[p,x]\phi&=&[p,i\hbar{d\over dp}]\phi=i\hbar\left(p{d\over dp}\...
...=i\hbar(-\phi)={\hbar\over i}\phi\\
{[p,x]}&=&{\hbar\over i}\\
\end{eqnarray*}

The commutator is the same in any representation.

\* [Example:](http://quantummechanics.ucsd.edu/ph130a/130_notes/node115.html#example:comEt) Compute the commutator .\*   
\* [Example:](http://quantummechanics.ucsd.edu/ph130a/130_notes/node116.html#example:comEx) Compute the commutator .\*   
\* [Example:](http://quantummechanics.ucsd.edu/ph130a/130_notes/node117.html#example:compxn) Compute the commutator .\*   
\* [Example:](http://quantummechanics.ucsd.edu/ph130a/130_notes/node118.html#example:comlxly) Compute the commutator of the angular momentum operators .\*

Buktikan bahwa E and X adalah Komutator

\begin{eqnarray*}[E,x]\psi(x,t)&=&\left(i\hbar{\partial\over \partial t}x-xi\hba...
...ial t}-i\hbar x{\partial\over \partial t}\right)\psi(x,t) =0 \\
\end{eqnarray*}

**=============================**

**Commutator of \bgroup\color{black}$E$\egroupand \bgroup\color{black}$t$\egroup**

Again use the crutch of keeping a wave function on the right to avoid mistakes.

\begin{eqnarray*}[E,t]\psi(x,t)&=&\left(i\hbar{\partial\over \partial t}t-ti\hba...
...rtial\over \partial t}\right)\psi(x,t)\\
&=&i\hbar\psi(x,t)\\
\end{eqnarray*}

Removing the wave function, we have the commutator.

\begin{displaymath}\bgroup\color{black} [E,t]=i\hbar \egroup\end{displaymath}

===================================

**Commutator of \bgroup\color{black}$E$\egroupand \bgroup\color{black}$x$\egroup**

Again use the crutch of keeping a wave function on the right to avoid mistakes.

\begin{eqnarray*}[E,x]\psi(x,t)&=&\left(i\hbar{\partial\over \partial t}x-xi\hba...
...ial t}-i\hbar x{\partial\over \partial t}\right)\psi(x,t) =0 \\
\end{eqnarray*}



Since .

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**Commutator of \bgroup\color{black}$L_x$\egroupand \bgroup\color{black}$L_y$\egroup**

Angular momentum is defined by

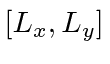
\begin{displaymath}\bgroup\color{black} \vec{L}=\vec{r}\times\vec{p} .\egroup\end{displaymath}

So the components of angular momentum are

\begin{displaymath}\bgroup\color{black} L_z=xp_y-yp_x \egroup\end{displaymath}

\begin{displaymath}\bgroup\color{black} L_x=yp_z-zp_y \egroup\end{displaymath}

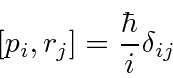
\begin{displaymath}\bgroup\color{black} L_y=zp_x-xp_z .\egroup\end{displaymath}

We wish to compute which has all the coordinates and momenta in it.

The only operators that do not commute are the coordinates and their conjugate momenta.

\begin{displaymath}\bgroup\color{black} [x,y]=0 \egroup\end{displaymath}

\begin{displaymath}\bgroup\color{black} [p_x,p_y]=0 \egroup\end{displaymath}

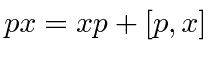


So now we just need to compute.

\begin{eqnarray*}[L_x,L_y]&=&[yp_z-zp_y,zp_x-xp_z]\\
&=&[yp_z,zp_x]-[yp_z,xp_z...
...-0-0+x[z,p_z]p_y\\
&=&{\hbar\over i}(yp_x-xp_y)=i\hbar L_z \\
\end{eqnarray*}

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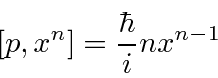
**Commutator of \bgroup\color{black}$p$\egroupand \bgroup\color{black}$x^n$\egroup**

We can use the commutator to help us. Remember that

\begin{eqnarray*}[p,x^n]&=&px^n-x^np\\
&=&(px)x^{n-1}-x^np\\
&=&xpx^{n-1}+[p...
...p,x]x^{n-1}-x^np\\
&=&n[p,x]x^{n-1}=n{\hbar\over i}x^{n-1} \\
\end{eqnarray*}

It is usually not wise to use the differential operators and a wave function crutch to compute commutators like this one. **Use the known basic commutators when you can.**Nevertheless, we can compute it that way.

\begin{displaymath}\bgroup\color{black}[p,x^n]\psi={\hbar\over i}{\partial\over\...
...{\partial\over\partial x}\psi={\hbar\over i}nx^{n-1}\psi\egroup\end{displaymath}



It works pretty well for this particular case, but not if I have \bgroup\color{black}$p$\egroupto some power.

==================================================

## Commutators of \bgroup\color{black}$A$\egroup, \bgroup\color{black}$A^\dagger$\egroupand \bgroup\color{black}$H$\egroup

We will use the commutator between \bgroup\color{black}$A$\egroupand \bgroup\color{black}$A^\dagger$\egroupto solve the HO problem. The operators are defined to be

\begin{eqnarray*}
A &=&\left(\sqrt{m\omega\over2\hbar}x+i{p\over\sqrt{2m\hbar\om...
...t{m\omega\over2\hbar}x-i{p\over\sqrt{2m\hbar\omega}}\right) .\\
\end{eqnarray*}

The **commutator** is

\begin{eqnarray*}[A,A^\dagger]&=&{m\omega\over2\hbar}[x,x]+{1\over 2m\hbar\omega...
...\\
&=&{i\over 2\hbar}(-[x,p]+[p,x])={i\over \hbar}[p,x]=1 .\\
\end{eqnarray*}

Lets use this simple commutator

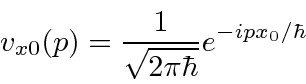
\begin{displaymath}\bgroup\color{black} [A,A^\dagger]=1 \egroup\end{displaymath}

to compute **commutators with the Hamiltonian**. This is easy if \bgroup\color{black}$H$\egroupis written in terms of \bgroup\color{black}$A$\egroupand \bgroup\color{black}$A^\dagger$\egroup.

\begin{eqnarray*}[H,A]&=&\hbar\omega[A^\dagger A,A]=\hbar\omega[A^\dagger,A]A=-\...
...er]=\hbar\omega A^\dagger[A,A^\dagger]=\hbar\omega A^\dagger \\
\end{eqnarray*}

## Operators in Momentum Space

If we want to work in **momentum space**, we need to look at the states of definite position to find our operators. The state (in momentum space) with definite position \bgroup\color{black}$x_0$\egroupis



The operators are

|  |
| --- |
| \bgroup\color{black}$\displaystyle x^{(op)}=i\hbar{\partial\over \partial p}$\egroup |

and

\begin{displaymath}\bgroup\color{black}p^{(op)}=p .\egroup\end{displaymath}

The \bgroup\color{black}$(op)$\egroupnotation used above is usually dropped. If we see the variable \bgroup\color{black}$p$\egroup, use of the operator is implied (except in state written in terms of \bgroup\color{black}$p$\egrouplike ).