18.1 Contour Integrals

DEFINITION 18.1 **Contour Integral**

Let f be defined at points of a smooth curve C given by $z = x(t) + iy(t),\ a \le t \le b$. The **contour integral** of f along C is

$$\int_C f(z)\,dz = \lim_{\|\Delta z_k\| \to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k \qquad (1)$$

THEOREM 18.1 **Evaluation of a Contour Integral**

If f is continuous on a smooth curve C given by $z(t) = x(t) + iy(t),\ a \le t \le b$, then

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt \qquad (3)$$

Example 1

Evaluate $\int_C \bar{z}\,dz$

where C is given by $x = 3t,\ y = t^2, -1 \le t \le 4.$

Solution

$$z(t) = 3t + it^2,\ z'(t) = 3 + 2it$$

$$f(z(t)) = \overline{3t + it^2} = 3t - it^2$$

Thus, $\int_C \bar{z}\,dz = \int_{-1}^{4} (3t - it^2)(3 + 2it)\,dt$

$$= \int_{-1}^{4} (2t^3 + 9t)\,dt + i\int_{-1}^{4} 3t^2\,dt = 195 + 65i$$

Example 2

Evaluate $\oint_C \frac{1}{z} dz$

where C is the circle $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.

Solution

$$z(t) = \cos t + i \sin t = e^{it}, \quad z'(t) = ie^{it}$$

$$f(z) = \frac{1}{z} = e^{-it}$$

Thus, $\oint_C \frac{1}{z} dz = \int_0^{2\pi} e^{-it} ie^{it} dt = 2\pi i$

THEOREM 18.2 Properties of Contour Integrals

Suppose f and g are continuous in a domain D and C is a smooth curve lying entirely in D. Then:

(i) $\int_C k f(z)\,dz = k\int_C f(z)\,dz$, k a constant

(ii) $\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$

(iii) $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$, where C is the union of the smooth curve C_1 and C_2.

(iv) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$, where $-C$ denotes the curve having the opposite orientation of C.

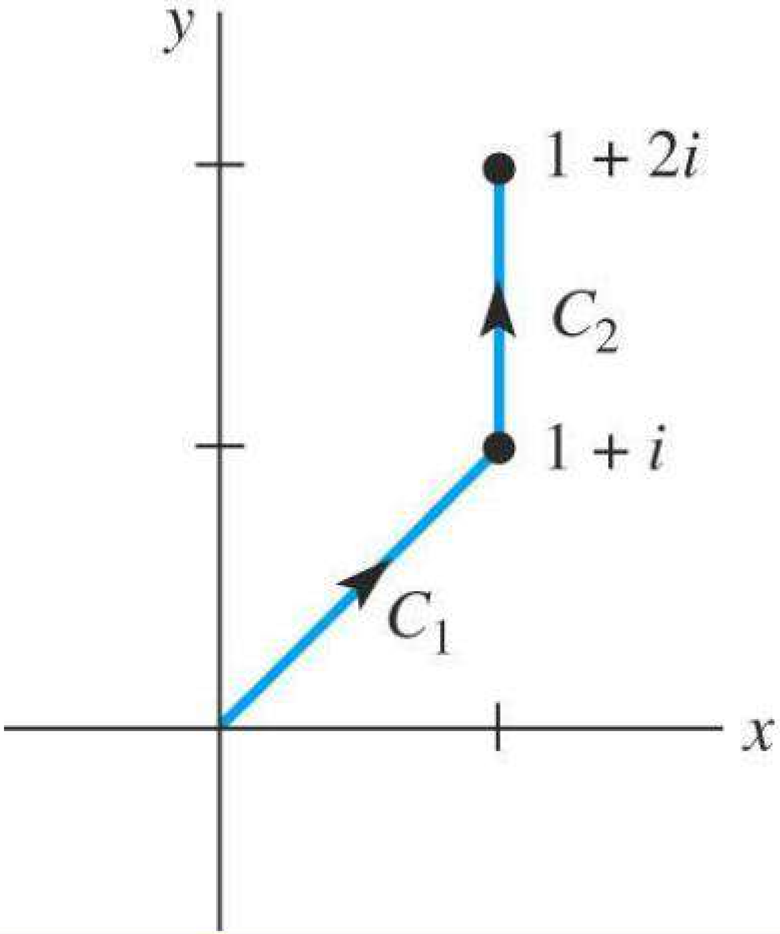
Example 3

Evaluate $\int_C (x^2 + iy^2)\,dz$

where C is the contour in Fig 18.1.

Solution

Fig 18.1



Example 3 (2)

We have

$$\int_C (x^2 + iy^2)dz = \int_{C_1} (x^2 + iy^2)dz + \int_{C_2} (x^2 + iy^2)dz$$

Since C_1 is defined by $y = x$, then $z(x) = x + ix$, $z'(x) = 1 + i$, $f(z(x)) = x^2 + ix^2$, and

$$\int_{C_1} (x^2 + iy^2)dz = \int_0^1 (x^2 + ix^2)(1 + i)dx$$

$$= (1 + i)^2 \int_0^1 x^2 dx = \frac{2}{3} i$$

Example 3 (3)

The curve C_2 is defined by $x = 1$, $1 \le y \le 2$. Then $z(y) = 1 + iy$, $z'(y) = i$, $f(z(y)) = 1 + iy^2$. Thus

$$\int_{C_2} (x^2 + iy^2)dz = \int_1^2 (1 + iy^2)idy$$

$$= -\int_1^2 y^2 dy + i\int_1^2 dy = -\frac{7}{3} + i$$

Finally, $\int_C (x^2 + iy^2)dz = \frac{2}{3}i + (-\frac{7}{3} + i) = -\frac{7}{3} + \frac{5}{3}i$

THEOREM 18.3 **A Bounding Theorem**

If f is continuous on a smooth curve C and if $|f(z)| \leq M$ for all z on C, then $\left|\int_C f(z)dz\right| \leq ML$, where L is the length of C.

- This theorem is sometimes called the ML-inequality

Example 4

Find an upper bound for the absolute value of

$$\oint_C \frac{e^z}{z+1}\,dz$$

where C is the circle $|z| = 4$.

Solution

Since $|z + 1| \geq |z| - 1 = 3$, then

$$\left|\frac{e^z}{z+1}\right| \leq \frac{|e^z|}{|z|-1} = \frac{|e^z|}{3} \tag{5}$$

Example 4 (2)

In addition, $|e^z| = e^x$, with $|z| = 4$, we have the maximum value of x is 4. Thus (5) becomes

$$\left|\frac{e^z}{z+1}\right| \leq \frac{e^4}{3}$$

Hence from Theorem 18.3,

$$\left|\oint_C \frac{e^z}{z+1}\,dz\right| \leq \frac{8\pi e^4}{3}$$

18.2 Cauchy-Goursat Theorem

Cauchy's Theorem

Suppose that a function f is analytic in a simply connected domain D and that f' is continuous in D. Then for every simple closed contour C in D,

$$\oint_C f(z)dz = 0$$

This proof is based on the result of Green's Theorem.

$$\oint_C f(z)dz$$

$$= \oint_C u(x, y)\,dx - v(x, y)\,dy + i\oint_C v(x, y)\,dx + u(x, y)\,dy$$

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA \qquad (1)$$

- Now since f is analytic, the Cauchy-Riemann equations imply the integral in (1) is identical zero.

THEOREM 18.4 **Cauchy-Goursat Theorem**

Suppose a function f is a analytic in a simply connected domain D. Then for every simple closed C in D,

$$\oint_C f(z)\,dz = 0$$

- Since the interior of a simple closed contour is a simply connected domain, the Cauchy-Goursat Theorem can be stated as

  If f is analytic at all points within and on a simple closed contour C,

$$\oint_C f(z)\,dz = 0 \qquad (2)$$

Example 1

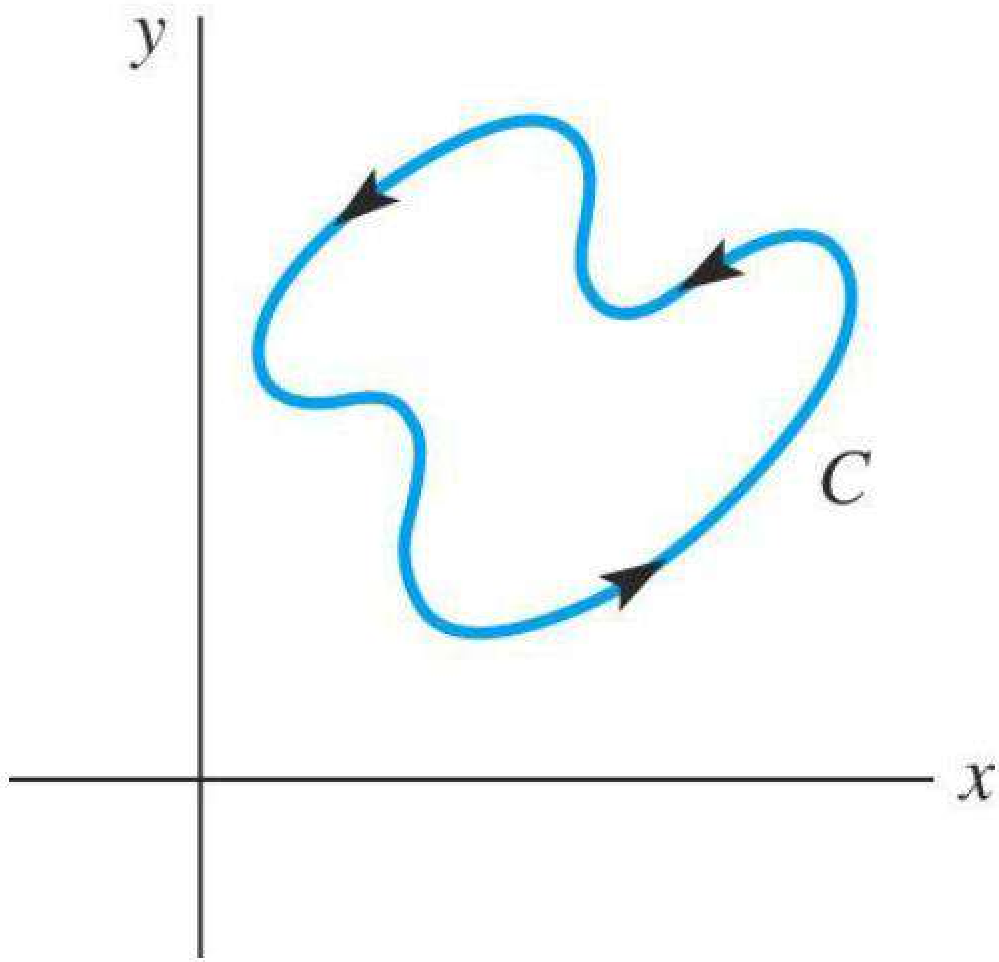
Evaluate $\oint_C e^z dz$

where C is shown in Fig 18.9.

Solution

The function e^z is entire and C is a simple closed contour. Thus the integral is zero.

Fig 18.9



Example 2

Evaluate $\oint_C \frac{dz}{z^z}$

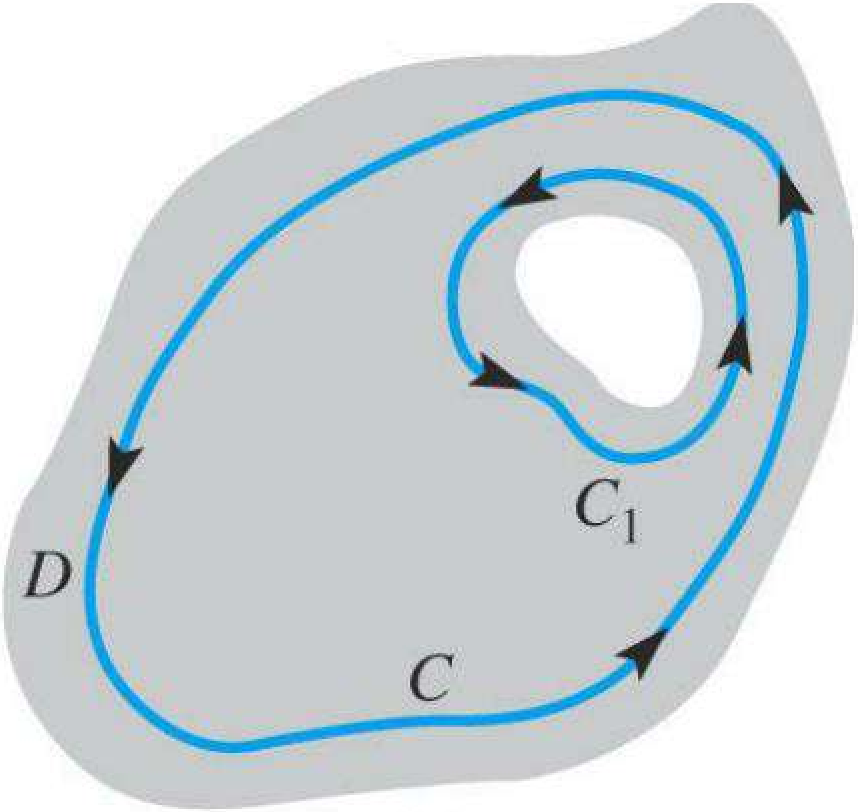
where C is the ellipse $(x-2)^2 + (y-5)^2/4 = 1$.

Solution

We find that $1/z^2$ is analytic except at $z = 0$ and $z = 0$ is not a point interior to or on C. Thus the integral is zero.

Cauchy-Goursat Theorem for Multiply Connected Domains

- Fig 18.11(a) shows that C_1 surrounds the "hole" in the domain and is interior to C.



(a)

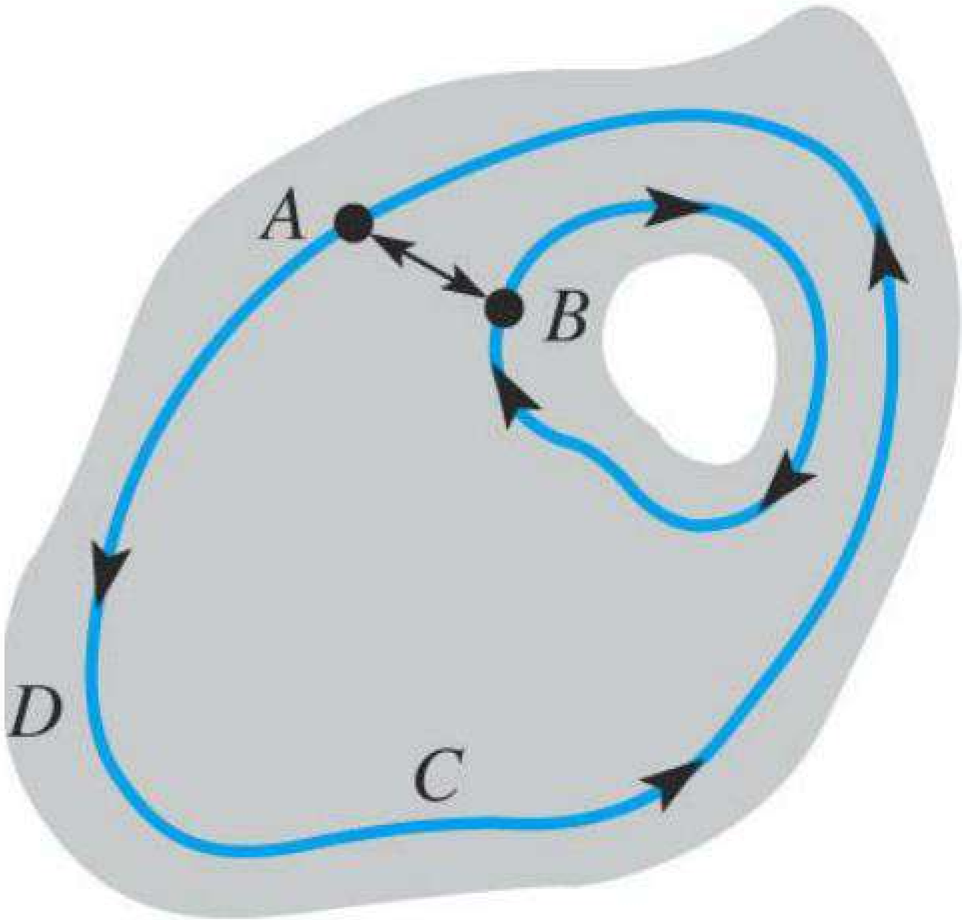
- Suppose also that f is analytic on each contour and at each point interior to C but exterior to C_1. When we introduce the cut AB shown in Fig 18.11(b), the region bounded by the curves is simply connected. Thus from (2)

$$\oint_C f(z)\,dz + \oint_{C_1} f(z)\,dz = 0$$

and

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz \qquad (3)$$

Fig 18.11 (b)



(b)

Example 4

Evaluate $\oint_C \frac{dz}{z-i}$

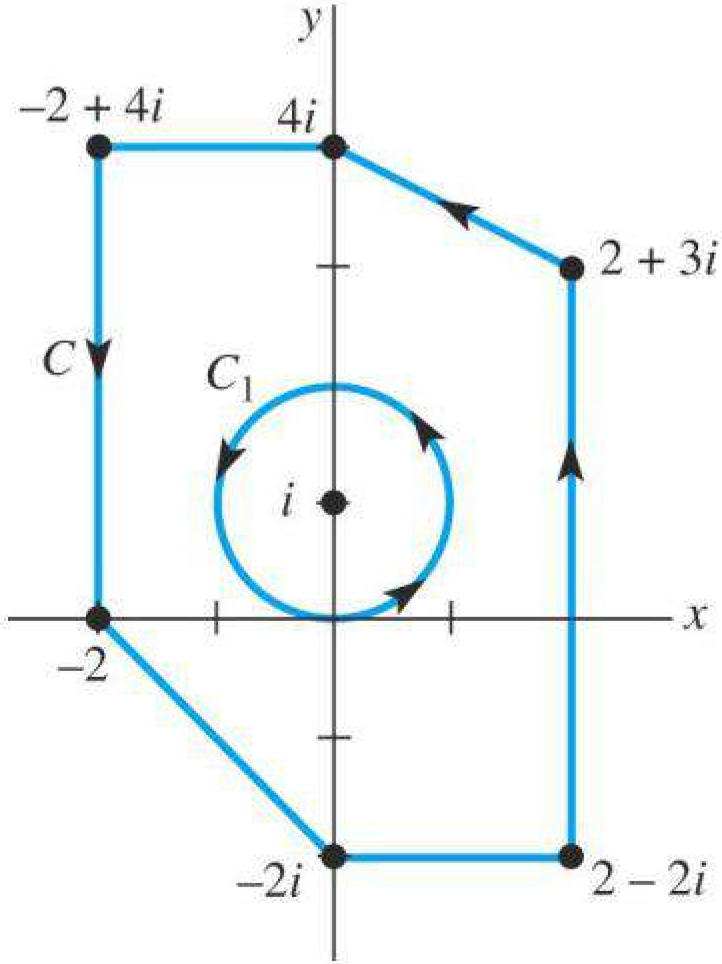
where C is the outer contour in Fig 18.12.

Solution

From (3), we choose the simpler circular contour C_1: $|z - i| = 1$ in the figure. Thus $x = \cos t$, $y = 1 + \sin t$, $0 \le t \le 2\pi$, or $z = i + e^{it}$, $0 \le t \le 2\pi$. Then

$$\oint_C \frac{dz}{z-i}\,dz = \oint_{C_1} \frac{dz}{z-i}\,dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = i\int_0^{2\pi} dt = 2\pi i$$

Fig 18.12



- The result in Example 4 can be generalized. We can show that if z_0 is any constant complex number interior to any simple closed contour C, then

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i, & n=1 \\ 0, & n \text{ an integer} \neq 1 \end{cases} \qquad (4)$$

Example 5

Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}dz$

where C is the circle $|z - 2| = 2$.

Solution

$$\frac{5z+7}{z^2+2z-3} = \frac{3}{z-1} + \frac{2}{z+3}$$

and so

$$\oint_C \frac{5z+7}{z^2+2z-3}dz = 3\oint_C \frac{dz}{z-1} + 2\oint_C \frac{dz}{z+3} \qquad (5)$$

Example 5 (2)

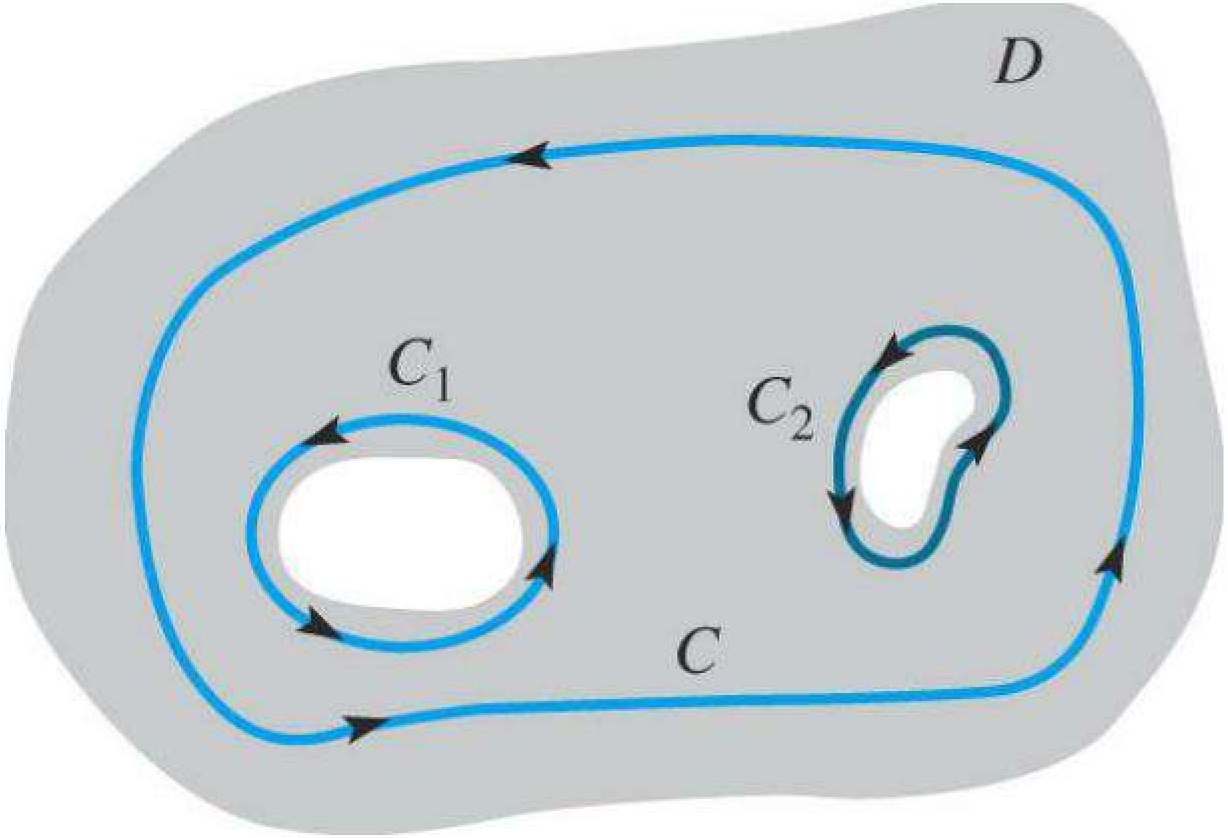
Since $z = 1$ is interior to C and $z = -3$ is exterior to C, we have

$$\oint_C \frac{5z+7}{z^2+2z-3}\,dz = 3(2\pi i) + 2(0) = 6\pi i$$

Fig 18.13

- See Fig 18.13. We can show that

$$\oint_C f(z)\, dz = \oint_{C_1} f(z)\, dz + \oint_{C_2} f(z)\, dz$$



THEOREM 18.5

Cauchy-Goursat Theorem for Multiply Connected Domain

Suppose C, C_1, …, C_n are simple closed curves with a positive orientation such that C_1, C_2, …, C_n are interior to C but the regions interior to each C_k, $k = 1, 2, \ldots, n$, have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the C_k, $k = 1, 2, \ldots, n$, then

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz \qquad (6)$$

Example 6

Evaluate $\oint_C \frac{dz}{z^2+1}$

where C is the circle $|z| = 3$.

Solution

$$\frac{1}{z^2+1} = \frac{1/2i}{z-i} - \frac{1/2i}{z+i}$$

$$\oint_C \frac{dz}{z^2+1} = \frac{1}{2i} \oint_C \left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz$$

Example 6 (2)

We now surround the points $z = i$ and $z = -i$ by circular contours C_1 and C_2. See Fig 18.14, we have

$$\oint_C \frac{dz}{z^2+1}$$

$$= \frac{1}{2i}\oint_{C_1}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]dz + \oint_{C_2}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]dz \qquad (7)$$

$$= \frac{1}{2i}\oint_{C_1}\frac{dz}{z-i} - \frac{1}{2i}\oint_{C_1}\frac{dz}{z+i} + \frac{1}{2i}\oint_{C_2}\frac{dz}{z-i} - \frac{1}{2i}\int_{C_2}\frac{dz}{z+i}$$

Since $\oint_{C_1}\frac{dz}{z-i}i = 2\pi i,\quad \oint_{C_2}\frac{dz}{z+i}i = 2\pi i$

thus (7) becomes zero.

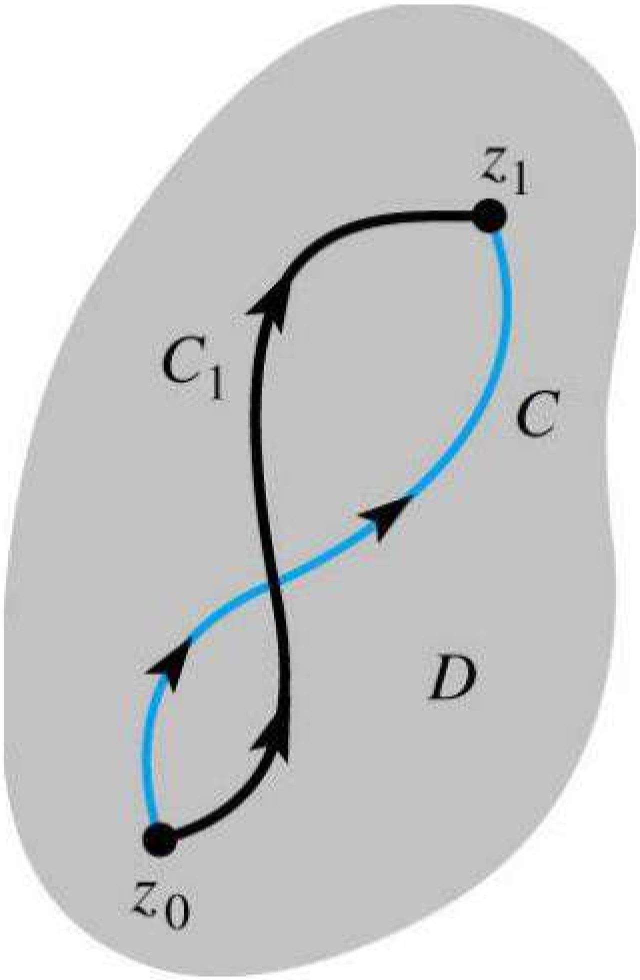
18.3 Independence of Path

DEFINITION 18.2 **Independence of the Path**

Let z_0 and z_1 be points in a domain D. A contour integral $\oint_C f(z)\,dz$ is said to be **independent of the path** if its value is the same for all contours C in D with an initial point z_0 and a terminal point z_1.

- See Fig 18.19.

Fig 18.19



- Note that C and C_1 form a closed contour. If f is analytic in D then

$$\int_C f(z)\,dz + \int_{-C_1} f(z)\,dz = 0 \qquad (2)$$

Thus

$$\int_C f(z)\,dz = \int_{-C_1} f(z)\,dz \qquad (3)$$